Simple Linear Regression Model With Periodically-Correlated Errors

By

Nour Hamad Suleiman Abu-Afouna

Supervisor

Dr. Abdullah A. Smadi

Co-Supervisor

Dr. Mohammed H. AL-Haj Ebrahem

Program: Statistics

July 7, 2011

Simple Linear Regression Model With Periodically-Correlated Errors

By

Nour Hamad Suleiman Abu-Afouna

B.Sc. Statistics, Yarmouk University, (2007)

A thesis submitted in partial fulfillment of requirements for the degree of Master of Science in the Department of Statistics, Faculty of Science, Yarmouk University, Irbid, Jordan

Approved by:

Dr. Abdullah A. Smadi.. Chairman

Associate Professor of Statistics, Yarmouk University

Dr. Omar M. Eidous .... Member

Associate Professor of Statistics, Yarmouk University

Dr. Hilmi F. Kittani .... Member

Associate Professor of Statistics, Hashemite University

July 7, 2011

إهداء إلى القلب النابض في صدري والفكر الدائم في ذهني

وابتسامة حياتي وقدوتي وقنديل ظلامي ونور أيامي والدي

الحبيب (رحمه الله)

ACKNOWLEDGMENT

First of all, I want to thank ALLAH who gave me courage and patience to accomplish this thesis, guide me and show me the way in the hard times.

I am heartily thankful to my supervisor Dr. Abdullah A. Smadi for his invaluable assistance, encouragement, advice, guidance, dedication, support and enormous patience from the initial to the final level which enabled me to develop an understanding of the thesis.

Also, I would like to show my gratitude to my co-supervisor Dr. Mohammed H. AL-Haj Ebrahem for his reviewing and supporting this thesis.

Of course, I will not forget to thank the examining committee and to staff member at the department of statistics for their guidance all throughout the period of graduate study.

Finally, I thanks to my precious family specially my beloved father (may God have mercy on him), my mother, my brothers, my sisters, and all their children, specially the cute little angel shaker. I am indebted to my best friend Areen Al-Quraan for his helping, continuous encouragement and support to pass the tough times. Also, I wish to thank everyone who helped me to complete this thesis.

TABLE OF CONTENTS

LIST OF TABLES viii

LIST OF FIGURES x

ABSTRACT xi

ABSTRACT (In Arabic) xiii

LIST OF SYMBOLS xiv

CHAPTER ONE: INTRODUCTION

1.1 Introduction 1

1.2 The ARIMA and PARMA models 3

1.3 Literature review 6

1.4 The problem of our research and objectives 7

1.5 Overview 8

CHAPTER TWO: THE POWER OF DURBIN-WATSON TEST FOR ARMA MODELS

2.1 Introduction 10

2.2 Time series regression models 10

2.3 Dynamic regression models 13

2.4 The Durbin-Watson test 13

2.5 Power of D-W test with errors following some ARMA models 16

2.6 Results and discussion 24

CHAPTER THREE: SIMPLE LINEAR REGRESSION MODEL WITH ERRORS FOLLOWING PAR(1)

3.1 Introduction 26

3.2 Simple linear regression model with errors following PAR(1)... 26

3.3 Estimation in $PAR_{\omega}(1)$ processes 27

3.4 Properties of least squares estimators for various models of errors 29

3.5 Power of D-W test with errors following various PAR(1) models 39

3.6 Discussion 42

CHAPTER FOUR: GENERALIZATION OF COCHRANE-ORCUTT PROCEDURE

4.1 Introduction 43

4.2 Cochrane-Orcutt procedure 43

4.3 Generalization of Cochrane-Orcutt procedure for errors following PAR(1) 45

4.4 An illustration of the proposed method 50

4.5 Bias and MSE of the proposed estimator using Monte-Carlo simulation 52

CHAPTER FIVE: APPLICATION TO REAL DATA

5.1 Introduction 55

5.2 The data 55

5.3 Methodology and analysis 56

CHAPTER SIX: CONCLUSIONS AND FUTURE WORK

6.1 Introduction 60

6.2 Conclusions 60

6.3 Future work 61

REFERENCES 62

LIST OF TABLES

Table Number		Page
2.1	The ACF of some ARMA models	15
2.2	Models of Case 2 with the same ρ_1	16
2.3	The first three autocorrelations for the models in Case (1)	18
2.4	The power of D-W test, errors following white-noise model (Case 1)	18
2.5	The power of D-W test, errors following AR(1) model (Case 1)	18
2.6	The power of D-W test, errors following MA(1) model (Case 1)	19
2.7	The power of D-W test, errors following ARMA(1,1) model (Case 1)	19
2.8	The power of D-W test for AR(1) model for n=100 (Case 2)	20
2.9	The power of D-W test for MA(1) model for n=100 (Case 2)	20
2.10	The power of D-W test for ARMA(1,1) model for n=100 (Case 2)	21
3.1	The relative efficiency of $\hat{\beta}_0$ and $\hat{\beta}_1$ under AR(1) model with respect to the WN model	35
3.2	The relative efficiency of $\hat{\beta}_0$ and $\hat{\beta}_1$ under $PAR_4(1)$ model with respect to the WN model	35
3.3	The relative efficiency of $\hat{\beta}_0$ and $\hat{\beta}_1$ under $PAR_4(1)$ model with respect to the AR(1) model	36

Table Number		Page
3.4	The power of D-W test, errors following $PAR_4(1)$ model (Model(1))	40
3.5	The power of D-W test, errors following $PAR_4(2)$ (Model(2)) and $PAR_4(2,1,0,2)$ (Model(3))	40
4.1	The Bias and MSE (in brackets) of estimates β_0 and $\beta 1$	52

LIST OF FIGURES

Figure Number **Page**

2.1	The residual plot of autocorrelated residuals	14
2.2	The power of D-W test for AR(1) model (Case 1) with n=30, 50, 100	21
2.3	The power of D-W test for MA(1) model (Case 1) with n=30, 50, 100	22
2.4	The power of D-W test for models: AR(1) , MA(1) and ARMA(1,1) models in Case 2 with equal ρ_1	22
3.1	The relative efficiency of $\hat{\beta}_0$ and $\hat{\beta}_1$ under AR(1) model with respect to the WN model	36
3.2	The relative efficiency of $\hat{\beta}_0$ and $\hat{\beta}_1$ under $PAR_4(1)$ model with respect to the WN model	37
4.1	The time series plot of Y(t)	49
5.1	The time series plot of quarterly U.S. airline passenger miles (in millions)	54
5.2	Parallel box plot of residuals by quarter	55
5.3	Scatter-plots of residuals of consecutive quarters	56

ABSTRACT

Abu-Afouna, Nour hamad Suleiman. Simple Linear Regression Model With Periodically-Correlated Errors. Master of Science Thesis, Department of Statistics, Yarmouk University, 2011. (Supervisor: Dr. Abdullah A. Smadi), (Co-Supervisor: Dr. Mohammed H. AL-Haj Ebrahem).

Regression analysis is a very important area in statistics for both theoretical and applied disciplines. In this thesis, the simple linear regression model with autocorrelated errors is considered.

The main objective of this thesis is the studying of simple linear regression model with periodically-correlated errors. Instead of assuming that the errors follow the autoregressive model of order one (AR(1)) which is traditionally assumed for autocorrelated errors, we assume that the errors follow the periodic autoregressive model of order one (PAR(1)) in this study. This model is useful for modeling periodically autocorrelated errors. In particular, it is expected to be useful when the data are seasonal.

In this research we have studied the estimation of parameters of the simple linear regression model when the errors follow PAR(1). Besides, the relative efficiency of these estimates are obtained and compared with the estimates for the white noise and AR(1) models.

The power of the Durbin-Watson test is also investigated through Monte-Carlo simulation for errors following AR(1) and PAR(1) models.

Finally, the remedial measure of autocorrelated errors known as Cochrane-Orcutt procedure is extended to the case of periodically autocorrelated errors. Using

Monte-Carlo simulation, we found that the estimates for the Cochrane-Orcutt procedure dominate the least squares estimates. Also, a real application is provided.

Key Words: Simple Linear Regression Model, Periodically-Correlated Errors, Cochrane-Orcutt procedure, Relative efficiency, Monte-Carlo simulation.

الملخص

أبو عفونة، نور حمد سليمان. نموذج الانحدار الخطي البسيط مع أخطاء مرتبطة دورياً. رسالة ماجستير في العلوم، قسم الإحصاء، جامعة اليرموك. 2011 (المشرف: الدكتور عبدالله أحمد الصمادي)، (المشرف المساعد: الدكتور محمد حسن الحاج إبراهيم).

تحليل الانحدار هو مجال هام في الإحصاء لكل من التخصصات النظرية والتطبيقية. في هذه الرسالة تم دراسة نموذج الانحدار الخطي البسيط مع وجود الأخطاء المرتبطة ذاتياً.

إن الهدف الرئيسي من هذا البحث هو دراسة نموذج الانحدار الخطي البسيط مع وجود أخطاء مرتبطة دوريا. بدلاً من الافتراض التقليدي بأن الأخطاء المرتبطة عادةً تتبع نموذج الانحدار الذاتي من الرتبة الأولى (AR(1))، فإننا افترضنا في هذا البحث أن الأخطاء تتبع نموذج الانحدار الذاتي الدوري من الرتبة الأولى (PAR(1)). هذا النموذج مفيد لوصف الأخطاء المرتبطة دورياً خصوصاً عندما تكون البيانات موسميه.

في هذا البحث درسنا تقدير معلمات نموذج الانحدار الخطي البسيط عندما تتبع الأخطاء نموذج PAR(1). بالإضافة إلى حساب الكفاءة النسبية لهذه التقديرات ومقارنتها بالتقديرات المقابلة لها في حالة نماذج WNو AR(1).

وتم أيضا إيجاد قوه اختبار ديربن-واتسون باستخدام أسلوب المحاكاة للأخطاء التابعة لنموذجي AR(1) و PAR(1).

أخيرا تم تعميم أسلوب المعالجة للأخطاء المرتبطة المعروف باسم إجراء كوكران-أوركت على حالة الأخطاء المرتبطة دورياً. و باستخدام أسلوب المحاكاة وجدنا أن التقديرات بأجراء كوكران-أوركت المطُور يتفوق على تقديرات المربعات الصغرى في كل من التحيز ومتوسط مربع الأخطاء.

الكلمات المفتاحية: نماذج الانحدار الخطي البسيط، الأخطاء المرتبطة دورياً، إجراء كوكران-أوركت، الكفاءة النسبية، محاكاة مونتي كارلو.

LIST OF SYMBOLS

ACF	Autocorrelation function
ACVF	Autocovariance function
AR	Autoregressive
ARIMA	Autoregressive integrated moving average
ARMA	Autoregressive moving average
BLUE	Best linear unbiased estimators
D-W	Durbin-watson test
LS	Least squares
MA	Moving average
ML	Maximum likelihood
OLS	Ordinary least squares
PAR	Periodic autoregressive
PARMA	Periodic autoregressive moving average
SACF	Seasonal autocorrelation function
SACVF	Seasonal autocovariance function
SL	Simple linear
WN	White noise

CHAPTER 1

Introduction

1.1 Introduction

Regression analysis is a very important statistical method that investigates the relationship between a response variable, usually denoted as Y, and a set of other variables named as independent variables or predictors, usually denoted by X_1, X_2, ..., X_p. An important objective of the builting model is the prediction of Y for given values of the predictors.

The simple linear regression model is the simplest regression model in which we have only one predictor X. This model, which could be the most common in practice, written as:

$$Y_t = \beta_0 + \beta_1 X_t + \varepsilon_t, \quad t = 1, 2, ..., n \qquad (1.1)$$

where (X_t, Y_t) are the values of the predictor and response variables in the t^{th} trial, respectively, β_0 and β_1 are unknown parameters and ε_t's are usually assumed iid $N(0, \sigma^2)$ specially for inference purposes. The variable X is usually assumed fixed and non-random. For several predictors, the multiple linear regression model is written as:

$$Y_t = \beta_0 + \beta_1 X_{1t} + \beta_2 X_{2t} + ... + \beta_p X_{pt} + \varepsilon_t, \quad t = 1,2,...,n$$

where Y_t is the value of the response variable in the t^{th} trial, β_0, β_1, ..., β_p are unknown parameters, X_{1t}, X_{2t}, ..., X_{pt} are the values of the predictor variables in the t^{th} trial and ε_t is a sequence of uncorrelated random errors with mean zero and variance σ^2. For the estimation of regression model there are two common methods, the first is the least squares (LS) method, which relies on minimizing the sum of squares of errors. For the simple linear regression model (1.1), the LS estimators of β_0 and β_1 are:

$$\hat{\beta}_0 = \bar{Y} - \hat{\beta}_1 \bar{X}$$

and (1.2)

$$\hat{\beta}_1 = \frac{\sum_{t=1}^{n}(X_t - \bar{X})(Y_t - \bar{Y})}{\sum_{t=1}^{n}(X_t - \bar{X})^2}.$$

The second method is the Maximum Likelihood (ML) method, which relies on maximizing the likelihood function of β_0 and β_1 and we use this method when the functional form of the probability distribution of the error terms is specified. Under the normality assumption of errors, it can be shown that $\hat{\beta}_0$ and $\hat{\beta}_1$ above are also the ML estimators of β_0 and β_1 (Kutner et al., 2005, p. 31-32).

In turn, the fitted regression model is written as:

$$\hat{Y}_t = \hat{\beta}_0 + \hat{\beta}_1 X_t.$$

Thus, the estimated errors or residuals, denoted by e_t, are defined as:

$$e_t = Y_t - \hat{Y}_t, \; t = 1,2,\dots,n.$$

After fitting the regression model, an important step in model building and diagnosis is to check for the assumptions of the model, namely; independence, normality and constant variance of errors. The residuals of the fitted model play a primary role for this purpose. Several graphs and tests for residuals may be used to examine these assumptions. These techniques are usually known as residual analysis. In particular, to examine the independence of error terms we use the plots of residuals against time usually named as the residual plot. An important test specifically designed for testing the lack of randomness in residuals, is the Durbin-Watson test (D-W test). For a detailed account on various methods for the assessment of model assumptions see (Kutner et al., 2005, p. 103-115).

In this research we will study the power of the D-W test assuming that the errors are dependent (autocorrelated) exhibiting various time series models. Those models are explained in the next section. Therefore, we postpone the discussion of D-W test to the next chapter.

When the assumptions of the regression model are not valid there exist several remedial measures. When the error terms are correlated, a direct remedial measure is to work with a model that calls for correlated error terms. In the next chapters we will explain and elaborate on one of those methods named as Cochrane-Orcutt procedure. For more details on such issues see (Kutner et al., 2005, p. 127).

1.2 The ARIMA and PARMA models

The seasonal autoregressive integrated moving average (ARIMA) model is the well-known extension of the ordinary autoregressive moving average (ARMA) model that is suitable for seasonal time series. The seasonal-ARIMA model of orders (p, d, q) × $(P, D, Q)_\omega$ can be written as:

$$\Phi_P(B^{\omega})\phi_p(B)(1-B)^d(1-B^{\omega})^D X_t = \theta_0 + \Theta_Q(B^{\omega})\theta_q(B)a_t, \quad (1.3)$$

where $\phi_p(B)$, $\theta_q(B)$, $\Phi_P(B^{\omega})$ and $\Theta_Q(B^{\omega})$ are the ordinary AR, ordinary MA, seasonal AR and seasonal MA polynomials, respectively, B is the backshift operator, ω is the number of periods, d and D are the ordinary and seasonal differencing orders and $\{a_t\}$ is the white noise process with zero mean and variance σ_a^2 and θ_0 is constant (Box et al., 1994). The most common values of the period ω in practice are 4 or 12 corresponding to the quarterly or monthly time series, respectively. The pure autoregressive model is a special case of the ARIMA model where d, P, D, q and Q are set to zero in (1.3). In this research we are concerned with the following three models which are special cases of (1.3); namely; the autoregressive model of order 1 (AR(1)) given by :

$$X_t = C + \phi_1 X_{t-1} + a_t \ , \quad (1.4)$$

the moving average model of order 1 (MA(1)) given by:

$$X_t = C + a_t - \theta_1 a_{t-1} , \quad (1.5)$$

and the mixed autoregressive moving average model (ARMA(1,1)) given by:

$$X_t = C + \phi_1 X_{t-1} + a_t - \theta_1 a_{t-1} . \quad (1.6)$$

The periodic autoregressive moving-average PARMA model is an extension of the ordinary Box-Jenkins ARMA model which is suitable for modeling seasonal time series. The $PARMA_{\omega}(p(v), q(v))$ model is written as:

$$(1-\phi_1(v)B - \ldots - \phi_{p(v)}(v)B^{p(v)})(X_{k\omega+v} - \mu_v) = (1-\theta_1(v)B - \ldots - \theta_{q(v)}(v)B^{q(v)})a_{k\omega+v}, \quad (1.7)$$

where ω denotes the number of period, v=1, 2, ..., ω denotes the season, k denotes the year, $\{a_{k\omega+v}\}$ is a zero-mean white noise process with periodic variances $\sigma_a^2(v)$, p(v) is the AR order for season v and q(v) is the MA order for season v, μ_v is the mean of season v and $\phi_1(v)$, ..., $\phi_{p(v)}(v)$ and $\theta_1(v)$, ..., $\theta_{q(v)}(v)$ are the AR and MA parameters of season v, respectively. For more details on PARMA models, see Franses and Paap (2004).

The periodic autoregressive model (PAR) is a special case of the PARMA model. In equation (1.7) setting q(v) = 0 for each v=1,2,..., ω we get the equation of the $PAR_\omega(p(v))$ written as:

$$X_{k\omega+v}-\mu_v=\sum_{j=1}^{p(v)}\phi_j(v)(X_{k\omega+v-1}-\mu_{v-j})+a_{k\omega+v}. \qquad (1.8)$$

For example, the $PAR_\omega(1)$ model can be written as:

$$X_{k\omega+v}-\mu_v=\phi_1(v)(X_{k\omega+v-1}-\mu_{v-1})+a_{k\omega+v}. \qquad (1.9)$$

In fact, this equation can be written as ω equations. For instance, the zero-mean $PAR_4(1)$ can be written as:

$$X_{4k+1}=\phi_1(1)X_{4(k-1)+4}+a_{4k+1}$$

$$X_{4k+2}=\phi_1(2)X_{4k+1}+a_{4k+2}$$

$$X_{4k+3}=\phi_1(3)X_{4k+2}+a_{4k+3}$$

$$X_{4k+4}=\phi_1(4)X_{4k+3}+a_{4k+4}$$

which is periodic stationary if $\left|\prod_{v=1}^{4}\phi_1(v)\right|<1$ (Obeysekera and Sals, 1986).

PARMA models are not stationary in the ordinary weak sense. They are rather examined for a weaker type of stationarity named as periodic stationarity. This means that the mean and the variance of the time series is constant for each season and periodic with period ω and the auto-covariance function depends on the time lag and season only (Ula and Smadi, 1997).

If the period $\omega=1$, then the $PAR_{\omega}(p)$ model reduces to the ordinary AR(p) model. For more details on PARMA models see for example Pagano (1978), Tiao and Grupe (1980) and Franses and Paap (2004).

1.3 Literature review

The regression models with correlated errors have been studied by many authors. Mohammed and Ibazizen (2008) considered a Bayesian analysis of regression models with errors following first-order autoregressive model. Kayode (2008) studied the estimation of linear models when stochastic regressors that are correlated and the error terms are autocorrelated. Huitema and McKean (2007) studied a wide variety of ARMA models for the errors associated with time series regression models. Stocker (2007) investigated the OLS method for stationary dynamic regression models, where the errors follow a stationary ARMA process. Fried and Gather (2005) discussed the robust estimation of a linear trend when the noise follows an autoregressive process of first order. Yue and Koreisha (2004) studied asymptotic and finite-sample properties of predictors of regression models with autocorrelated errors. Gulhan and Pierre (2001) studied the efficiency analysis of many estimation procedures for linear models with autocorrelated errors. Smith et al. (1998) proposed the Bayesian approach for non-parametric estimation of regression models with autocorrelated errors. Ohtani (1990) examined the small-sample properties of the generalized least squares (GSL) estimators and tests of individual regression coefficients with autocorrelated errors.

1.4 The Problem of our research and objectives

In this research, we propose the idea of periodically autocorrelated errors in regression model. The D-W test is used to test if the errors are autocorrelated, i.e., the errors follow an AR(1) model. Here, we assume that the errors of the regression model follow the $PAR_\omega(1)$ model. Writing the time t as $k\omega+v$; the errors $\{\varepsilon_t\}$ follow:

$$\varepsilon_{k\omega+v} = \phi_1(v)\varepsilon_{k\omega+v-1} + a_{k\omega+v}, \qquad (1.10)$$

which is the same as (1.9) with $\mu_v=0$, $v=1, 2, \ldots, \omega$.

The essence of (1.10) is that the errors are periodically autocorrelated which means that the autocorrelations among successive errors changes from one season to another. For example, assuming that $\{Z_{12k+v}\}$ is a monthly time series, then periodic autocorrelation means that Corr(Z_{12k+v} , $Z_{12k+v-1}$) is not constant for $v=1, 2, \ldots, 12$. McLeod (1995) suggests a simple graphical method to detect periodic autocorrelations in time series. For instance, for monthly time series, he suggested to sketch the scatter plots of the time series values for (Feb vs Jan), (March vs Feb), ..., (Dec vs Nov). If those scatter plots exhibit different patterns of bivariate relationships, then this may lead to periodic autocorrelations in the time series.

In addition, McLeod (1995) proposed a test of periodically autocorrelated errors. McLeod proposed to apply this test on the residuals resulted from fitting seasonal ARIMA models for some seasonal time series. If significant, this test will indicate that the residuals are periodically autocorrelated and therefore a PARMA model should be considered for modeling this time series in place of the seasonal ARIMA model. This test will be used later on the errors of the regression model when Y_t is a seasonal time series.

In this research, we will consider the simple linear regression model (1.1) and assume that $\{Y_t\}$ is a seasonal time series with number of periods ω, X_t is assumed to be non-random and $\{\varepsilon_t\}$ follows a $PAR_\omega(1)$ model. Next, we interest in estimating the parameters β_0, β_1, $\sigma_a^2(\nu)$ and $\phi_1(\nu)$, $\nu=1, 2, \ldots, \omega$ of the regression model under this setting. We will also examine the behaviors (powers) of D-W test for various $PAR_\omega(1)$ models. For the latter objective, we will use Monte-Carlo simulation assuming that $\{\varepsilon_t\}$ follows the $PAR_\omega(1)$ model (1.10) by manipulating the values of parameters $\phi_1(1),\ldots,\phi_1(\omega)$ as well as $\sigma_a^2(\nu)$.

1.5 Overview

In this chapter, we have reviewed the regression analysis, the ARIMA model, the ordinary ARMA model and its extension PARMA model which suitable for modeling seasonal time series. In chapter two we discussed the time series and dynamic regression models and studied the Durbin-Watson test and its power for errors following the models WN, AR(1), MA(1) and ARMA(1,1).

In chapter three, we study the simple linear regression model with errors following $PAR_4(1)$ model. We estimated the variances and the seasonal autocorrelation function (SACF) for $PAR_\omega(1)$, then we used relative efficiency to compare between the least squares estimation for the parameters for the WN, AR(1) and $PAR_4(1)$ models. Using Monte-Carlo simulation technigue, we compute the power of Durbin-Watson test with errors following $PAR_4(1)$ model and applied the work to higher orders PAR model. The studying of the Cochrane-Orcutt procedure for AR(1) and the generalization the work for $PAR_4(1)$, then using Monte-Carlo simulation to compare between the least squares method and Cochrane-Orcutt method via bias and MSE are done in chapter four. In chapter five we applied the generalized Cochrane-Orcutt method on real time series data.

Finally, chapter six gives the results, conclusions and some ideas for future work.

CHAPTER 2

The Power of Durbin-Watson test for ARMA Models

2.1 Introduction

In this chapter we discuss time series regression models. Such models relate the dependent variable Y_t to functions of time. Also, we will study an important diagnostic-checking method of independence condition among errors using D-W test and study the power of this test with errors following some ARMA models using Monte-Carlo simulation.

2.2 Time series regression models

When a time series exhibits a deterministic trend, then the trend can be modeled using polynomial functions of time. In particular, the no trend, linear trend and quadratic trend models are famous special cases of this model.

We sometimes can describe a time series Y_t by using trend model. The trend model is defined as follows:

$$\begin{aligned} Y_t &= TR_t + \varepsilon_t \\ &= \beta_0 + \beta_1 t + \beta_2 t^2 + \ldots + \beta_p t^p + \varepsilon_t, \quad t=1,2,\ldots,n \end{aligned}$$

where Y_t is the value of the time series in period t, TR_t is the trend in time period t and ε_t is the error term in time period t. these models are found in (Bowerman et al., 2005, p.279).

When the time series exhibits deterministic seasonal variation, we often use the following additive model:

$$Y_t = TR_t + SN_t + \varepsilon_t,\ t = 1, 2, \ldots, n$$

where Y_t is the observed value of the time series in time period t, TR_t is the trend in time period t, SN_t is the seasonal factor in time period t and ε_t is the error term in time period t. One way to model seasonal patterns is to use dummy variables. Assuming that there are L seasons (months, quarters) per year, we state the seasonal factor SN_t as follows (Wei, 1990, p.159):

$$SN_t = \beta_{s1}X_{s1,t} + \beta_{s2}X_{s2,t} + \ldots + \beta_{s(L-1)}X_{s(L-1),t},\ t = 1,2,\ldots, n$$

where $X_{s1,t}, X_{s2,t}, \ldots, X_{s(L-1),t}$ are dummy variables. This model assumes constant variance and seasonality of the time series. If the seasonality is increasing we usually use $ln(Y_t)$ in place of Y_t.

An alternative method for modeling seasonality is to use a linear combination of trigonometric functions. Two such models that can be useful when modeling constant seasonal variation are:

$$Y_t = TR_t + \beta_2 Sin\left(\frac{2\pi t}{L}\right) + \beta_3 Cos\left(\frac{2\pi t}{L}\right) + \varepsilon_t$$

and

$$Y_t = TR_t + \beta_2 Sin\left(\frac{2\pi t}{L}\right) + \beta_3 Cos\left(\frac{2\pi t}{L}\right) + \beta_4 Sin\left(\frac{4\pi t}{L}\right) + \beta_5 Cos\left(\frac{4\pi t}{L}\right) + \varepsilon_t$$

Extra Sine and Cosine terms can also be added for modeling more complicated seasonality (Bowerman et al., 2005).

The trend component (TR_t) in the above two models is usually modeled by linear or quadratic functions of t. Besides, the parameters of all models above are usually estimated using the ordinary least squares method assuming that ε_t are iid $N(0, \sigma_\varepsilon^2)$.

2.3 Dynamic regression models

Assuming that $\{(X_t,Y_t);\ t \in Z\}$ is a bivariate stochastic process from which a realization (bivariate time series) of length n is observed then the simple regression model (1.1) along its standard assumptions is usually invalid, simply because X_t is random. In fact, in this case, this model is known as the stochastic (dynamic) regression model which has different assumptions and implications than the ordinary regression model. However, in the literature of time series analysis, the simple linear regression model (1.1) is still applied under several situations. The first one is that assuming that X_t is a non-random (covariate) time series. For instance, X_t may be a deterministic function of time such as deterministic trend component or seasonal dummies. Another case is that viewing the regression model as a conditional relationship given X_t. In many applications such covariate time series $\{X_t\}$ is called a leading factor of Y_t (Cryer and Chan, 2008, p.265).

2.4 The Durbin-Watson test

We have seen in chapter one that the ordinary regression model has several assumptions including the assumption of independence among errors. An important diagnostic-checking method of this assumption is the D-W test which is very common specially when the data are related to time as in the time series regression models explained above.

When dealing with the D-W test, we use the term autocorrelated errors in place of the term dependent errors. Thus the null and alternative hypotheses in D-W test are:

H_0: Errors are not autocorrelated vs H_1: Errors are autocorrelated

In fact, the D-W test assumes that the errors follow the first-order autoregressive (AR(1)) model, written as

$$\varepsilon_t = \phi\varepsilon_{t-1} + a_t, \quad t = 1, 2, \ldots, n \qquad (2.1)$$

where $\{a_t\}$ is a white noise process assumed to be iid $N(0, \sigma_a^2)$ independent of ε_t and ϕ is the AR parameter with $|\phi| < 1$. Therefore, if $\phi = 0$ then $\varepsilon_t = a_t$ and the errors are considered un-autocorrelated (independent). It can be shown that the first order autcorrelation of $\{\varepsilon_t\}$ is,

$$\rho = Corr(\varepsilon_t, \varepsilon_{t-1}) = \phi.$$

In fact, the D-W test can be used to carry-out three different tests:

Test 1: Test of positive autocorrelation

$$H_0: \phi = 0 \quad \text{vs} \quad H_1: \phi > 0$$

Test 2: Test of negative autocorrelation

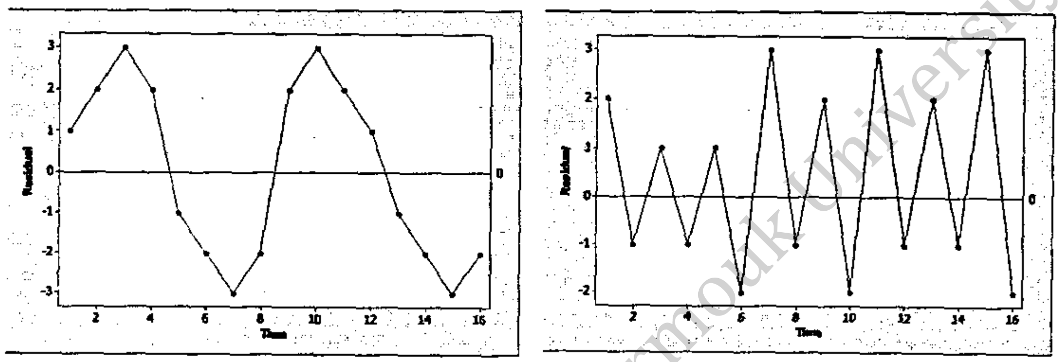
$$H_0: \phi = 0 \quad \text{vs} \quad H_1: \phi < 0$$

Test 3: Test of autocorrelation

$$H_0: \phi = 0 \quad \text{vs} \quad H_1: \phi \neq 0.$$

Based on the fitted regression model, the residual plot is usually a useful method for detecting autocorrelation among residuals. Figures (2-a) and (2-b) are two residual plots that indicate respectively positively autocorrelated and negatively autocorrelated residuals. It is found in practice that the positive autocorrelation is the most frequent type, so that the most common D-W test is Test-(1) above (Bowerman et al., 2005).

Figure 2.1: The residual plot of autocorrelated residuals



(a) Positively autocorrelated residuals (b) Negatively autocorrelated residuals

The D-W test statistic is (Durbin and Watson, 1950):

$$D = \frac{\sum_{t=2}^{n}(e_t - e_{t-1})^2}{\sum_{t=1}^{n} e_t^2},$$

which is compared with special critical values according to the types of the regression model and test. Exact critical values of the D-W test are difficult to obtain, but Durbin and Watson (Kutner et al., 2005) have obtained lower and upper bounds d_L and d_U such that a value of D outside these bounds leads to a definite decision. The decision rule for testing positively autocorrelated is:

If $D > d_U$ conclude H_0, if $D < d_L$ conclude H_1 otherwise the test is inconclusive.

Tables of the D-W test containing the limits d_U and d_L are available in most regression and time series books for selected the values of α, n the number of observation and k the number of predictors. For example see (Kutner et al., 2005, p.487).

2.5 Power of D-W test with errors following some ARMA models

The D-W test with errors following ARMA model have been studied by many authors. Blattberg (1973) studied the power of the D-W statistic for non-first order serial correlation alternatives. Schmidt and Guilkey (1975) compared the powers of the D-W and Geary tests.

In this section we will use Monte-Carlo simulation to examine the power of D-W test for errors following various ARMA model. We will consider the WN, AR(1), MA(1) and ARMA(1,1) processes. The main objective of considering various ARMA models is to investigate the power of D-W test for various autocorrelations-structures. Table (2.1) reviews the ACF formulas for some selected ARMA models. For a detailed account on ACF of various ARMA models and its properties, see Cryer and Chan, 2008.

Table (2.1): The ACF of some ARMA models

Number of Model	Model	ACF
1	WN	$\rho_k = \begin{cases} 1, & k=0 \\ 0, & k \neq 0 \end{cases}$
2	**AR(1)**	$\rho_k = \begin{cases} 1, & k=0 \\ \phi^k, & k \geq 1 \end{cases}$
3	**MA(1)**	$\rho_k = \begin{cases} 1, & k=0 \\ \frac{-\theta}{1+\theta^2}, & k=1 \\ 0, & k \geq 2 \end{cases}$
4	**ARMA(1,1)**	$\rho_k = \begin{cases} 1, & k=0 \\ \frac{(1-\theta\phi)(\phi-\theta)}{1-2\theta\phi+\theta^2}\phi^{k-1}, & k \geq 1 \end{cases}$

Therefore, we will carry out our study based on the following two cases:

Case 1: The models:

Model 1: WN process, iid N(0, σ^2) with σ^2=1.

Model 2: AR(1) with ϕ = -0.8, -0.2, 0.2, 0.8.

Model 3: MA(1) with θ = -0.8, -0.2, 0.2, 0.8.

Model 4: ARMA(1,1) with (ϕ , θ) = {-0.8, -0.2, 0.2, 0.8}.

Case 2: The AR(1), MA(1) and ARMA(1,1) models with parameters as defined in Table (2.2) below.

Table (2.2): Models of Case 2 with the same ρ_1

AR(1)	MA(1)	ARMA(1,1)		ρ_1
ϕ	θ	ϕ	θ	
0.000000000	0.000000000	0.1	0.1	0.000000000
0.075675676	-0.076114092	-0.8	-0.9	0.075675676
0.098000000	-0.098959716	-0.1	-0.2	0.098000000
0.140000000	-0.142857143	0.9	0.8	0.140000000
0.184000000	-0.190690788	-0.2	-0.4	0.184000000
0.204210526	-0.213520707	0.3	0.1	0.204210526
0.246000000	-0.263017889	-0.3	-0.6	0.246000000
0.260000000	-0.280449500	0.8	0.6	0.260000000
0.309677419	-0.346955888	0.4	0.1	0.309677419
0.360000000	-0.425036024	0.7	0.4	0.360000000
0.400000000	-0.500000000	0.8	0.5	0.400000000
0.440000000	-0.596620814	0.6	0.2	0.440000000
0.471641791	-0.708177196	0.7	0.3	0.471641791
0.490909091	-0.825179509	0.2	-0.4	0.490909091

In Case (1) our objective is to investigate the effect of the main parameters in each model on the power of the D-W test. We have selected the values of the parameters arbitrarily but satisfying stationarity and invertibility conditions.

In Case (2) we have selected a list of different AR(1), MA(1) and ARMA(1,1) models such that they agree on the value of the first lag autocorrelation (ρ_1). Table (2.2) shows the parameters of the selected models. It can be seen that each row in the table defines three different models but have the same ρ_1 value (given in the last column). The selection of these models is mainly based on the formulas of the theoretical ACF of various models given in Table (2.1). Besides, we have restricted ourselves to models with $\rho_1 < 0.5$ because for the invertible MA(1) model it is known that $|\rho_1| < 0.5$.

As far as the simulation-work is concerned; 2000 repetitions each of realization length n (30, 50, 100) pairs of data (X,Y) are simulated as follows:

(1) Set the predictor values X from 0 to 10 of n equally distant values, denote as X_1, X_2, …, X_n.

(2) Generate a realization of length n $\{\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n\}$ from one of the models above.

(3) Compute $Y_i = 2+5X_i+\varepsilon_i$; i=1, 2, …, n.

(4) Fit the simple linear regression model using the least squares method and compute the residuals.

(5) Compute the D-W test-statistic (of either type; positive autocorrelated or two-sided) and then compute the p-value of each test.

(6) Compute the proportion of significant tests (p-value ≤ 0.05) for all repitions which estimates the power of the test .

The simulation results for Case (1) are presented in Tables (2.4)-(2.7) and Figures (2.2)-(2.3), while the results for Case (2) are presented in Tables (2.8)-(2.10) and Figure (2.4).

Table (2.3): The first three autocorrelations for the models in Case (1)

Model			ρ_1	ρ_2	ρ_3
WN			0.0000	0.0000	0.0000
AR(1)	ϕ	-0.8	-0.8000	0.6400	-0.5120
		-0.2	-0.2000	0.0400	-0.0080
		0.2	0.2000	0.0400	0.0080
		0.8	0.8000	0.6400	0.5120
MA(1)	θ	-0.8	0.4878	0.0000	0.0000
		-0.2	0.1923	0.0000	0.0000
		0.2	-0.1923	0.0000	0.0000
		0.8	-0.4878	0.0000	0.0000
ARMA(1,1)	(ϕ,θ)	$\phi=\theta$	0.0000	0.0000	0.0000
		(-0.8,-0.2)	-0.7000	0.5600	-0.4480
		(-0.8,0.2)	-0.8529	0.6823	-0.5458
		(-0.8,0.8)	-0.8986	0.7189	-0.5751
		(-0.2,-0.8)	0.3818	-0.0763	0.0152
		(-0.2,0.2)	-0.3714	0.0742	-0.0148
		(-0.2,0.8)	-0.5918	0.1183	-0.0236
		(0.2,-0.8)	0.5918	0.1183	0.0236
		(0.2,-0.2)	0.3714	0.0742	0.0148
		(0.2,0.8)	-0.3818	-0.0763	-0.0152
		(0.8,-0.8)	0.8986	0.7189	0.5751
		(0.8,-0.2)	0.8529	0.6823	0.5458
		(0.8,0.2)	0.7000	0.5600	0.4480

Table (2.4): The power of D-W test, errors following white-noise model (Case 1)

n	Two-sided	Positive
30	0.0485	0.0465
50	0.0500	0.0455
100	0.0605	0.0550
500	0.0540	0.0530
1000	0.0530	0.0480

Table (2.5): The power of D-W test, errors following AR(1) model (Case 1)

		Two-sided			Positive autocorrelated		
		n			n		
ϕ	ρ_1	30	50	100	30	50	100
-0.8	-0.8000	0.9750	0.9995	1.0000	0.0000	0.0000	0.0000
-0.2	-0.2000	0.1505	0.2510	0.5025	0.0080	0.0030	0.0000
0.2	0.2000	0.1580	0.2605	0.5150	0.2445	0.3795	0.6105
0.8	0.8000	0.9605	0.9990	1.0000	0.9740	0.9995	1.0000

Table (2.6): The power of D-W test, errors following MA(1) model (Case 1)

θ	ρ_1	Two-sided			Positive autocorrelated		
		n			n		
		30	50	100	30	50	100
-0.8	0.4878	0.7760	0.9630	1.0000	0.8670	0.9835	1.0000
-0.2	0.1923	0.1610	0.2385	0.4745	0.2560	0.3325	0.5960
0.2	-0.1923	0.1360	0.2345	0.4675	0.0050	0.0030	0.0000
0.8	-0.4878	0.6535	0.9335	0.9995	0.0000	0.0000	0.0000

Table (2.7): The power of D-W test, errors following ARMA(1,1) model (Case 1)

ϕ	θ	ρ_1	Two-sided			Positive autocorrelated		
			n			n		
			30	50	100	30	50	100
-0.8	-0.8	0.0000	0.0510	0.0430	0.0480	0.0495	0.0515	0.0535
	-0.2	-0.7000	0.9110	0.9945	1.0000	0.0000	0.0000	0.0000
	0.2	-0.8529	0.9910	1.0000	1.0000	0.0000	0.0000	0.0000
	0.8	-0.8986	0.9990	1.0000	1.0000	0.0000	0.0000	0.0000
-0.2	-0.8	0.3818	0.5530	0.8040	0.9855	0.6870	0.8855	0.9960
	-0.2	0.0000	0.0455	0.0500	0.0500	0.0500	0.0375	0.0485
	0.2	-0.3714	0.4360	0.6925	0.9605	0.0005	0.0000	0.0000
	0.8	-0.5918	0.8755	0.9900	1.0000	0.0000	0.0000	0.0000
0.2	-0.8	0.5918	0.9305	0.9940	1.0000	0.9625	0.9985	1.0000
	-0.2	0.3714	0.4420	0.7390	0.9645	0.5820	0.8280	0.9820
	0.2	0.0000	0.0465	0.0460	0.0495	0.0565	0.0480	0.0485
	0.8	-0.3818	0.3820	0.7010	0.9760	0.0000	0.0000	0.0000
0.8	-0.8	0.8986	0.9995	1.0000	1.0000	1.0000	1.0000	1.0000
	-0.2	0.8529	0.9900	1.0000	1.0000	0.9985	1.0000	1.0000
	0.2	0.7000	0.8005	0.9750	1.0000	0.8590	0.9865	1.0000
	0.8	0.0000	0.0520	0.0465	0.0450	0.0555	0.0570	0.0480

Table (2.8): The power of D-W test for AR(1) model for n=100 (Case 2)

ϕ	ρ_1	Two-sided	Positive autocorrelated
0.000000000	0.000000000	0.0525	0.0480
0.075675676	0.075675676	0.0965	0.1815
0.098000000	0.098000000	0.1540	0.2575
0.140000000	0.140000000	0.2625	0.3810
0.184000000	0.184000000	0.4410	0.5525
0.204210526	0.204210526	0.4920	0.6515
0.246000000	0.246000000	0.6665	0.7700
0.260000000	0.260000000	0.7020	0.8235
0.309677419	0.309677419	0.8505	0.9150
0.360000000	0.360000000	0.9445	0.9675
0.400000000	0.400000000	0.9710	0.9885
0.440000000	0.440000000	0.9850	0.9965
0.471641791	0.471641791	0.9945	0.9970
0.490909091	0.490909091	0.9990	0.9990

Table (2.9): The power of D-W test for MA(1) model for n=100 (Case 2)

θ	ρ_1	Two-sided	Positive autocorrelated
0.000000000	0.000000000	0.0525	0.0525
-0.076114092	0.075675676	0.1180	0.1935
-0.098959716	0.098000000	0.1600	0.2480
-0.142857143	0.140000000	0.2755	0.3810
-0.190690788	0.184000000	0.4510	0.5585
-0.213520707	0.204210526	0.5140	0.6540
-0.263017889	0.246000000	0.6825	0.7790
-0.280449500	0.260000000	0.7740	0.8275
-0.346955888	0.309677419	0.8835	0.9415
-0.425036024	0.360000000	0.9695	0.9855
-0.500000000	0.400000000	0.9915	0.9955
-0.596620814	0.440000000	0.9995	1.0000
-0.708177196	0.471641791	1.0000	1.0000
-0.825179509	0.490909091	0.9995	1.0000

Table (2.10): The power of D-W test for ARMA(1,1) model for n=100 (Case 2)

ϕ	θ	ρ_1	Two-sided	Positive autocorrelated
0.1	0.1	0.000000000	0.0480	0.0530
-0.8	-0.9	0.075675676	0.0980	0.1615
-0.1	-0.2	0.098000000	0.1615	0.2260
0.9	0.8	0.140000000	0.2055	0.2820
-0.2	-0.4	0.184000000	0.4495	0.5850
0.3	0.1	0.204210526	0.4900	0.6330
-0.3	-0.6	0.246000000	0.7115	0.8145
0.8	0.6	0.260000000	0.5870	0.6760
0.4	0.1	0.309677419	0.8360	0.9050
0.7	0.4	0.360000000	0.8745	0.9260
0.8	0.5	0.400000000	0.8885	0.9350
0.6	0.2	0.440000000	0.9825	0.9895
0.7	0.3	0.471641791	0.9815	0.9885
0.2	-0.4	0.490909091	0.9995	1.0000

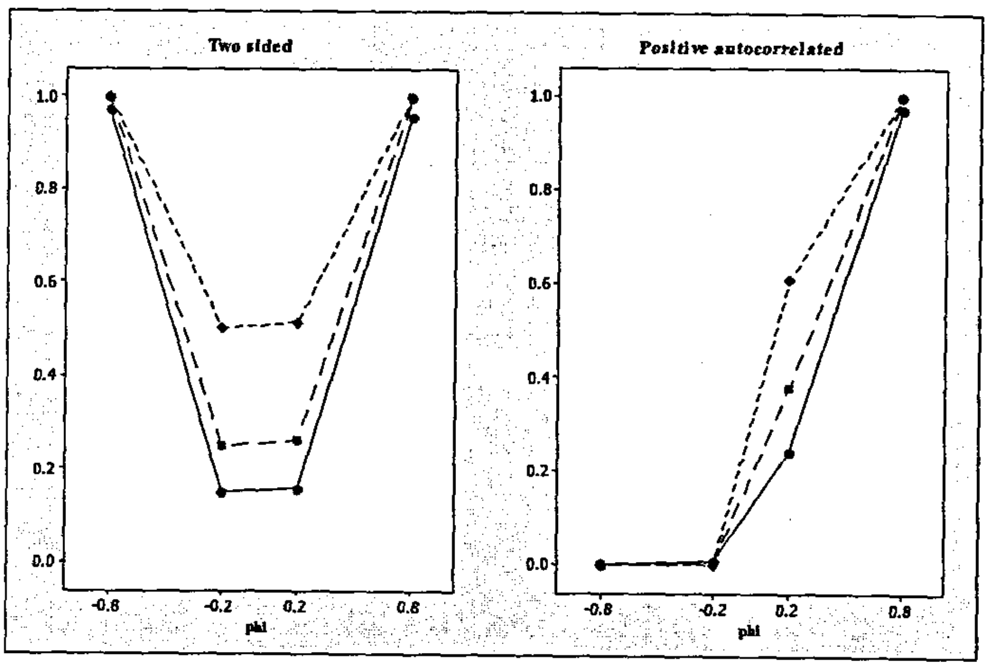


Figure (2.2): The power of D-W test for AR(1) model (Case 1) with n=30 (——), n=50 (– –), n=100 (- -)

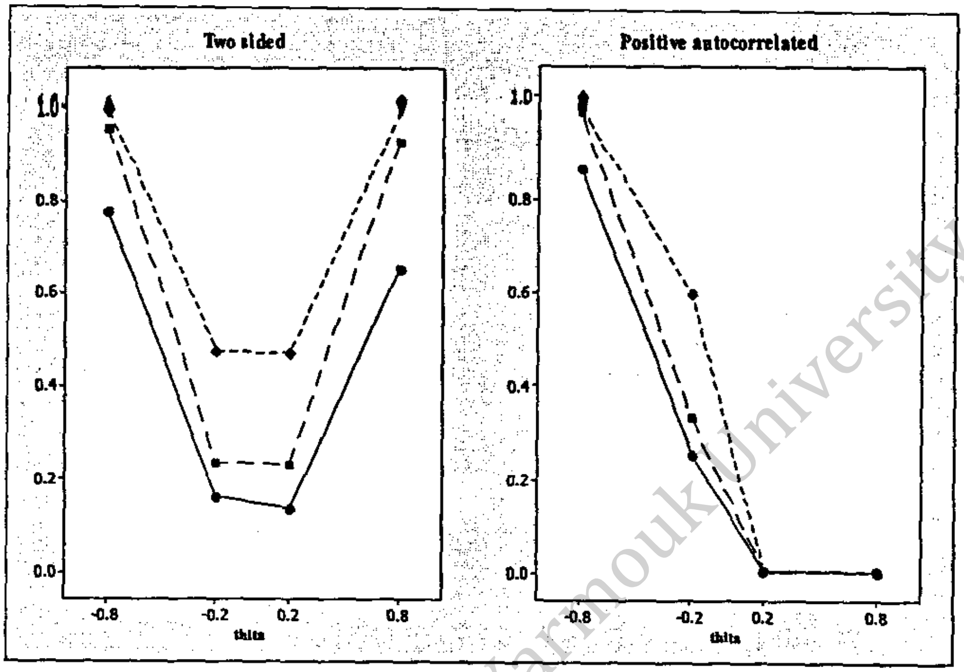


Figure (2.3): The power of D-W test for MA(1) model (Case 1) with n=30 (——), n=50 (– –), n= 100 (- -)

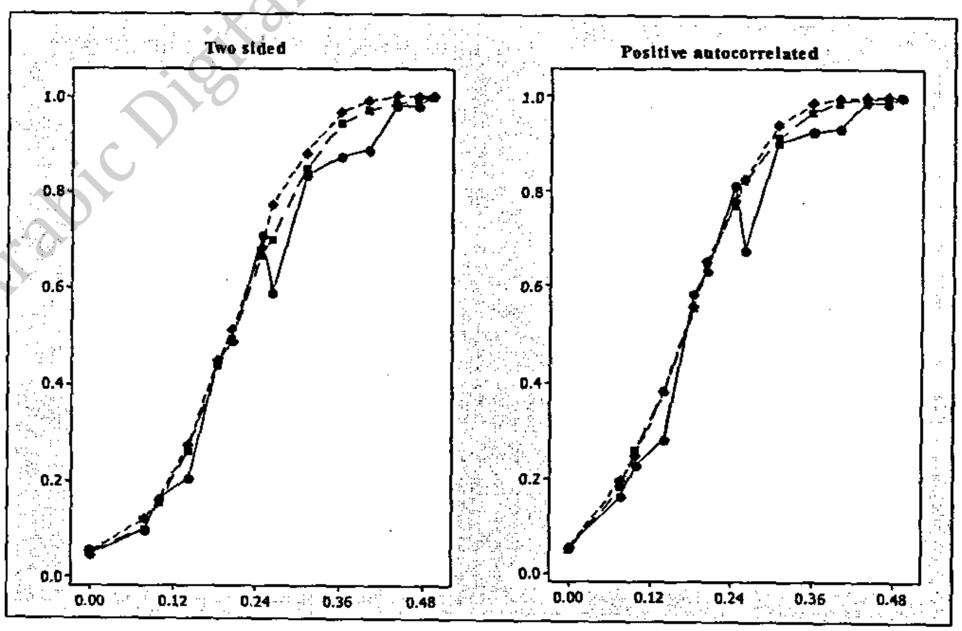


Figure (2.4): The power of D-W test for models: AR(1) (– –) , MA(1): (- -) and ARMA(1,1): (——) models in Case 2 with equal ρ_1 (x-axis)

2.6 Results and discussion

In view of the simulation results, we can see that in general the power of the D-W test increases when the realization length increases. In Table (2.4) the power of the D-W test is close to 0.05 for both types (positive or two-sided) which is expected since the model is WN. For the AR(1) model, Table(2.5) and Figures (2.2) show that the test of positive-autocorrelation are significant for the case with $\phi > 0$ with increasing power as ϕ approaches 1. However, for the two-sided test, the power is symmetric in view of ϕ with increasing power as $|\phi|$ approaches 1.

For the MA(1) model, Table (2.6) and Figure (2.3) show nearly similar results to the AR(1) model. That is, when the first autocorrelation (ρ_1) was positive, which corresponds to negative θ as seen in Tables (2.2) and (2.3), then both tests of two-sided and positive autocorrelation were significant. Also, for positive θ which corresponds to negative ρ_1, the two-sided test was significant but the test of positive autocorrelation was not.

The results of the ARMA(1,1) model that presented in Table (2.7) shows -in general- similar patterns as the previous models. The case of $\phi = \theta$ makes the ACF of the ARMA(1,1) model identical to the ACF of the WN process. This explains the power value of nearly 0.05 in Table (2.7) for these cases. Also, we can see from Table (2.3) that the largest (possible) value of ρ_1 occurs at $(\phi, \theta) = (0.8, -0.8)$ and the smallest (negative) value occurs at $(\phi, \theta) = (-0.8, 0.8)$. Therefore, it is seen from Table (2.7) that the D-W test of both types were highly significant for $(\phi, \theta) = (0.8, -0.8)$. For $(\phi, \theta) = (-0.8, 0.8)$ it is seen form Table (2.7) that the two-sided test was highly significant but the test of positive autocorrelation has a power close to zero.

It can also be seen from Figure (2.2), Figure (2.3) and Table (2.7) regardless of model type that the power of D-W test of either types were more stable for various values of n when ρ_1 is close to zero or $|\rho_1|$ is close to one. For moderate to small values of ρ_1 (as ρ_1=0.2), it can be seen that the power of D-W test of either types was highly affected by n.

The results of Case 2 which shown in Figure (2.4) indicates that there is a minor impact of the values of ϕ and θ on the type of model on the power of D-W test when ρ_1 is the same. When ρ_1 increases the power of the D-W test is also increasing for both the two sided and positive autocorrelated tests. A closer look at the power values shows that it was the largest for the MA(1) model. Also, the power values for the MA(1) and AR(1) models seem systematically increase as ρ_1 increases. However, for the ARMA(1,1) models, the power values were the smallest in almost all cases with some apparent fluctuations, specially for moderate values of ρ_1.

Finally, we close this section with two main points. The first is that the correlation structure of the error terms has a minor effect on the power of D-W test in simple LR model. We expect this result to be true for multiple LR model. The second point, which is a by-product of the first, is that the D-W test is blind for the autocorrelation type of errors. This means that, although the hypothesis of D-W test originally assumes an AR(1) model of the errors, the test if significant does not necessarily imply an AR(1) errors or it is uncapable for the identification of the model of the errors. A similar conclusion was reported by many researchers as for example (Blattberg, 1973, pp. 508-515).

CHAPTER 3

Simple Linear Regression Model with Errors Following PAR(1)

3.1 Introduction

In this chapter, we will study the idea of periodically autocorrelated errors in regression models. We will study the properties of least squares estimators when the error terms follow WN, AR(1) and PAR(1) models. Furthermore, a comparison between these models is carried out through relative efficiency (RE) by using simulation technique.

3.2 Simple linear regression model with errors following PAR(1)

Now, we consider the simple linear regression model (1.1) but assuming that the errors follow the zero-mean $PAR_{\omega}(1)$ model defined by (1.6). That is, writing t in (1.1) as $k\omega+v$, then the error terms $\{\varepsilon_t\}$ is modeled as:

$$\varepsilon_{k\omega+v} = \phi_1(v)\varepsilon_{k\omega+v-1} + a_{k\omega+v}, \qquad (3.1)$$

where ω denotes the number of periods, v=1,2,…,ω denotes the season, k denotes the year, $\{a_{k\omega+v}\}$ is a zero-mean white noise process with periodic variances $\sigma_a^2(v)$ and $\phi_1(v)$ is the AR parameter of season v with $\left|\prod_{v=1}^{\omega}\phi_1(v)\right|<1$.

The essence of (3.1) is that the errors are periodically autocorrelated which means that the autocorrelations among successive errors changes from one season to another. For example, Corr($\varepsilon_{k\omega+2}$, $\varepsilon_{k\omega+1}$) may differ from Corr($\varepsilon_{k\omega+3}$, $\varepsilon_{k\omega+2}$) although the two pairs of errors are both one-time lag distant.

We expect that such generalized regression model may apply for seasonal time series $\{Y_t\}$. Among others, McLeod (1995) and Frances and Paap (2004) proved that many seasonal time series may posses significant periodic autocorrelations. Besides, it is found that such periodicity is sustained even after fitting some seasonal models. In an earlier work, Tiao and Grupe (1980) called such patterns in autocorrelations as hidden periodicity that may not be captured by ordinary seasonal models.

The errors model in (3.1) reduces to the AR(1) model if $\phi_1(\nu)$ and $\sigma_a^2(\nu)$ are constants for all seasons ν.

In the next section we will review some results for the estimation of the $PAR_\omega(1)$ model. Later on, we will study the LS estimators for various models of errors including the PAR(1) model.

3.3 Estimation in $PAR_\omega(1)$ processes

The method of moments is one of the most common methods of estimation in statistical inference. It is also common in the context of time series analysis.

The seasonal autocorrelation function (SACF) depends on the time lag and season only and is defined as:

$$\rho_j(\nu) = Corr(X_{k\omega+\nu}, X_{k\omega+\nu-j}) = \frac{\gamma_j(\nu)}{\sqrt{\gamma_0(\nu)\gamma_0(\nu-j)}}, \tag{3.2}$$

where $\gamma_j(\nu) = Cov(X_{k\omega+\nu}, X_{k\omega+\nu-j})$ denotes the seasonal autocovariance function and $\gamma_0(\nu)$ denotes the variance of the process for season ν and time lag j=0,1,...

Based on an observed realization $Z_1, Z_2, \ldots, Z_{m\omega}$ the moment estimator of $\rho_k(\nu)$ is

$$r_k(\nu) = \frac{C_k(\nu)}{\sqrt{C_0(\nu)C_0(\nu-k)}}, \tag{3.3}$$

where

$$C_o(\nu)=\frac{\sum_{j=0}^{m-1}(Z_{j\omega+\nu}-\bar{Z}_\nu)^2}{m-1} \tag{3.4}$$

and

$$C_k(\nu)=\frac{\sum_{j=0}^{m-1}(Z_{j\omega+\nu}-\bar{Z}_\nu)(Z_{j\omega+\nu-k}-\bar{Z}_{\nu-k})}{m-1},$$

where $\bar{Z}_\nu$ is the sample mean of data in season ν and m is the number of years of data. It can be shown that $r_k(\nu)$ are asymptotically unbiased and consistent estimators (McLeod, 1995).

As far as the PAR$_\omega$(1) is considered, it can be proved that the first lag autocorrelations are given by:

$$\rho_1(\nu)=\phi_1(\nu)\sqrt{\frac{\gamma_0(\nu-1)}{\gamma_0(\nu)}}, \tag{3.5}$$

for ν=1,2,...,ω. Note that in this case the first order autocorrelation are not the same as AR parameters but a function of them. For the computation of $\rho_1(\nu)$, given $\phi_1(\nu)$ and $\sigma_a^2(\nu)$, (3.5) can be used along the fact that for the PAR$_\omega$(1) model:

$$\gamma_0(\nu)=(\phi_1(\nu))^2\gamma_0(\nu-1)+\sigma_a^2(\nu)\ ;\ \nu=1,\dots,\omega \tag{3.6}$$

This system of ω equations can be written as $A\Gamma_0=\Sigma$ where

$$A=\begin{bmatrix}1 & 0 & 0 & \dots & -\phi_1^2(1)\\ -\phi_1^2(2) & 1 & 0 & \dots & 0\\ 0 & -\phi_1^2(3) & 1 & \dots & 0\\ . & & & & \\ . & & & & \\ . & & & & \\ 0 & \dots & & -\phi_1^2(\omega) & 1\end{bmatrix},\ \Gamma_0=\begin{bmatrix}\gamma_0(1)\\ \gamma_0(2)\\ \gamma_0(3)\\ .\\ .\\ .\\ \gamma_0(\omega)\end{bmatrix},\ \Sigma=\begin{bmatrix}\sigma_a^2(1)\\ \sigma_a^2(2)\\ \sigma_a^2(3)\\ .\\ .\\ .\\ \sigma_a^2(\omega)\end{bmatrix}$$

As A is non-singular, we have

$$\Gamma_0=A^{-1}\Sigma. \tag{3.7}$$

For the estimation of $\phi_1(\nu)$ in the $PAR_\omega(1)$ model, the moment estimator of $\phi_1(\nu)$ is nearly the same as the least squares estimate obtained by regressing $Z_{k\omega+v}$ on $Z_{k\omega+v-1}$ in a season-wise manner. These estimators are also the conditional maximum likelihood estimators when the WN process is Gaussian (Smadi, 2005).

Thus, an alternative estimator of $\rho_1(v)$ for the $PAR_\omega(1)$ is obtained by using (3.5) and by replacing $\phi_1(\nu)$, $\gamma_0(\nu)$ and $\gamma_0(\nu-1)$ by their estimates.

3.4 Properties of least squares estimators for various models of errors

In chapter 1, we defined the simple linear regression model (1.1) with uncorrelated errors, mean zero and constant variance σ^2 for errors. We used the least squares (LS) method to estimate the regression model. It is known that the least squares estimators for the standard regression model are unbiased. They are in fact the best linear unbiased estimators (BLUE). More details about those estimators are found in (Kutner et al., 2005, p. 41-49).

Now, we will study the properties of least square estimators for the simple linear regression when the errors following the zero-mean AR(1) and the zero-mean $PAR_\omega(1)$ models.

Definition (3.1): Let $\{X_t\}$ be a stationary stochastic process. Then the autocovariance function (ACVF) of X_t is defined by:

$$\gamma_k = \text{Cov}(X_t, X_{t-k}) \; ; \; k = 0, 1, \ldots$$

For more details about stationary stochastic process and their ACVF, see for example (Wei, 1990)

Theorem (3.1): For the generalized linear regression model $Y_t = \beta_0 + \beta_1 X_t + \varepsilon_t$

with $\{\varepsilon_t\}$ being any stationary stochastic process with mean zero and ACVF γ_k, then the LS estimators $\hat{\beta}_0$ and $\hat{\beta}_1$ are unbiased and

$$Var(\hat{\beta}_0) = \sum_{t=1}^{n} c_t^2 \gamma_0 + 2 \sum_{t>j}\sum^{n} c_t c_j \gamma_{t-j} \tag{3.8a}$$

$$Var(\hat{\beta}_1) = \sum_{t=1}^{n} b_t^2 \gamma_0 + 2 \sum_{t>j}\sum^{n} b_t b_j \gamma_{t-j} \tag{3.8b}$$

with

$$c_t = \frac{1}{n} - b_t \bar{X} \tag{3.9a}$$

$$b_t = \frac{X_t - \bar{X}}{\sum_{t=1}^{n} (X_t - \bar{X})^2}. \tag{3.9b}$$

Proof: It can easily be shown that

$$\hat{\beta}_0 = \sum_{t=1}^{n} c_t Y_t \qquad \text{and} \qquad \hat{\beta}_1 = \sum_{t=1}^{n} b_t Y_t$$

where c_t and b_t are given by (3.9*a*) and (3.9b) respectively. Thus,

$$E(\hat{\beta}_0) = \sum_{t=1}^{n} c_t E(Y_t) = \sum_{t=1}^{n} c_t (\beta_0 + \beta_1 X_t) = \beta_0 \sum_{t=1}^{n} c_t + \beta_1 \sum_{t=1}^{n} c_t X_t.$$

Because, $\sum_{t=1}^{n} c_t = 1$ and $\sum_{t=1}^{n} c_t X_t = 0$ then $E(\hat{\beta}_0) = \beta_0$.

Similarly we can show $E(\hat{\beta}_1) = \beta_1$. The result in (3.8) is direct from the fact that $\hat{\beta}_0$ and $\hat{\beta}_1$ are linear function in $Y_1, Y_2, \ldots, Y_n$ with coefficients as in (3.9*a*) and (3.9b) respectively. Q. E. D.

Corollary (3.1): For the generalized regression model with $\{\varepsilon_t\}$ following the zero-mean AR(1) model, $\hat{\beta}_0$ and $\hat{\beta}_1$ are unbiased and the variance of $\hat{\beta}_0$ and $\hat{\beta}_1$ are:

$$\mathrm{Var}(\hat{\beta}_0) = \frac{\sigma_a^2}{(1-\phi^2)S_{XX}}\left[\left(\frac{S_{XX}}{n} + \bar{X}^2\right) + 2\sum_{t>j}^{n}\sum K_{tj}\phi^{t-j}\right] \tag{3.10}$$

$$\mathrm{Var}(\hat{\beta}_1) = \frac{\sigma_a^2}{(1-\phi^2)S_{XX}}\left[1 + \frac{2}{S_{XX}}\sum_{t>j}^{n}\sum M_{tj}\phi^{t-j}\right] \tag{3.11}$$

where $S_{XX} = \sum_{i=1}^{n}(X_i - \bar{X})^2$,

$$M_{tj} = (X_t - \bar{X})(X_j - \bar{X}) \tag{3.12}$$

and

$$K_{tj} = \frac{S_{XX}}{n^2} - \frac{\bar{X}}{n}\left((X_t - \bar{X}) + (X_j - \bar{X})\right) + \frac{\bar{X}^2}{S_{XX}}(X_t - \bar{X})(X_j - \bar{X}). \quad \text{Q. E. D.} \tag{3.13}$$

It can be seen easily that $Var(\hat{\beta}_0)$ and $Var(\hat{\beta}_1)$ above reduce to those of the standard regression model when $\phi = 0$; that is:

$$Var(\hat{\beta}_0) = \frac{\sigma^2}{S_{XX}}\left[\frac{S_{XX}}{n} + \bar{X}^2\right] \tag{3.14a}$$

and

$$Var(\hat{\beta}_1) = \frac{\sigma^2}{S_{XX}}. \tag{3.14b}$$

Theorem (3.2): For the generalized simple linear regression model with errors following a zero-mean PAR_ω (1) model, $\hat{\beta}_0$ and $\hat{\beta}_1$ are unbiased and the variances of $\hat{\beta}_0$ and $\hat{\beta}_1$ are given by:

$$Var(\hat{\beta}_0) = \sum_{v=1}^{\omega} \gamma_0(v) \sum_{k=1}^{m} c^2_{(k-1)\omega+v} + 2 \sum_{v=1}^{\omega} \sum_{k=1}^{m} \sum_{j=1}^{(k-1)\omega+v-1} c_{(k-1)\omega+v} c_j \gamma_{(k-1)\omega+v-j}(v) I((k-1)\omega + v - j \geq 1) \tag{3.15a}$$

$$Var(\hat{\beta}_1) = \sum_{v=1}^{\omega} \gamma_0(v) \sum_{k=1}^{m} b^2_{(k-1)\omega+v} + 2 \sum_{v=1}^{\omega} \sum_{k=1}^{m} \sum_{j=1}^{(k-1)\omega+v-1} b_{(k-1)\omega+v} b_j \gamma_{(k-1)\omega+v-j}(v) I((k-1)\omega + v - j \geq 1) \tag{3.15b}$$

where

$$c_{(k-1)\omega+v} = \frac{1}{n} - b_{(k-1)\omega+v} \overline{X} \tag{3.16a}$$

and

$$b_{(k-1)\omega+v} = \frac{X_{(k-1)\omega+v} - \overline{X}}{\sum_{l=1}^{n} (X_l - \overline{X})^2} \tag{3.16b}$$

Proof: The unbiasedness of the estimators $\hat{\beta}_0$ and $\hat{\beta}_1$ can be proved in a similar manner as Theorem (3.1). Now, we want to prove formulas (3.15*a*) and (3.15b). We know that

$$Var(\hat{\beta}_0) = \sum_{t=1}^{n} c_t^2 var(\varepsilon_t) + 2 \sum_{t>j}\sum^{n} c_t c_j Cov(\varepsilon_t, \varepsilon_j) \tag{3.17}$$

where c_t is as defined by (3.9*a*).

Now, assuming $n=m\omega$ where m is the number of years of data, then:

$$\mathrm{Var}(\hat{\beta}_0) = \sum_{\nu=1}^{\omega}\sum_{k=1}^{m} c^2_{(k-1)\omega+\nu}\gamma_0(\nu) + \sum_{\nu=1}^{\omega}\sum_{k=1}^{m}\sum_{j=1}^{(k-1)\omega+\nu-1} c_{(k-1)\omega+\nu}c_j\gamma_{(k-1)\omega+\nu-j}(\nu)\mathrm{I}((k-1)\omega+\nu-j\geq 1)$$

$$\mathrm{Var}(\hat{\beta}_0) = \sum_{\nu=1}^{\omega}\gamma_0(\nu)\sum_{k=1}^{m} c^2_{(k-1)\omega+\nu} + \sum_{\nu=1}^{\omega}\sum_{k=1}^{m}\sum_{j=1}^{(k-1)\omega+\nu-1} c_{(k-1)\omega+\nu}c_j\gamma_{(k-1)\omega+\nu-j}(\nu)\mathrm{I}((k-1)\omega+\nu-j\geq 1)$$

where I(A) is the indicator function. $Var(\hat{\beta}_1)$ in (3.15b) can be similarly derived. Q. E. D.

In fact, Theorem (3.2) can also be applied for all periodic-stationary processes with seasonal ACVF $\gamma_k(\nu)$.

The following theorem is useful for computation of the SACVF ($\gamma_k(\nu)$) for $\mathrm{PAR}_\omega(1)$ models needed for obtaining $Var(\hat{\beta}_0)$ and $Var(\hat{\beta}_1)$ when errors are PAR(1) as in Theorem (3.2).

Theorem (3.3): If $\{\varepsilon_t\}$ follows a periodic stationary $\mathrm{PAR}_\omega(1)$ model, then the SACVF at lag k; $\gamma_k(\nu)$ satisfies:

$$\begin{aligned}\gamma_k(\nu) &= \phi_1(\nu)\gamma_{k-1}(\nu-1)\\ &= \phi_1(\nu)\phi_1(\nu-1)\phi_1(\nu-2)\ldots\phi_1(\nu-k+1)\gamma_0(\nu-k).\end{aligned} \tag{3.18}$$

Proof: It is easy to show that for k=1,

$$\begin{aligned}\gamma_1(\nu) &= \mathrm{Cov}(\varepsilon_{k\omega+\nu}, \varepsilon_{k\omega+\nu-1})\\ &= \mathrm{Cov}(\phi_1(\nu)\varepsilon_{k\omega+\nu-1} + a_{k\omega+\nu}, \varepsilon_{k\omega+\nu-1})\\ &= \phi_1(\nu)\gamma_0(\nu-1)\end{aligned}$$

So that for k=2,

$$\gamma_2(\nu) = \text{Cov}(\varepsilon_{k\omega+\nu}, \varepsilon_{k\omega+\nu-2})$$
$$= \text{Cov}(\phi_1(\nu)\varepsilon_{k\omega+\nu-1} + a_{k\omega+\nu}, \varepsilon_{k\omega+\nu-2})$$
$$= \phi_1(\nu)\gamma_1(\nu-1)$$
$$= \phi_1(\nu)\phi_1(\nu-1)\gamma_0(\nu-2).$$

Similarly, we iterate k up to zero which gives (3.18). Q. E. D

Definition (3.2): (Rohatgi, 1984; p193) Let $\hat{\theta}_1$ and $\hat{\theta}_2$ be two estimators of θ. Then the relative efficiency of $\hat{\theta}_1$ with respect to $\hat{\theta}_2$ is:

$$RE(\hat{\theta}_1, \hat{\theta}_2) = \frac{MSE(\hat{\theta}_2)}{MSE(\hat{\theta}_1)}.$$

In addition, if $\hat{\theta}_1$ and $\hat{\theta}_2$ are unbiased then

$$RE(\hat{\theta}_1, \hat{\theta}_2) = \frac{Var(\hat{\theta}_2)}{Var(\hat{\theta}_1)}. \qquad (3.19)$$

Therefore and since $\hat{\beta}_0$ and $\hat{\beta}_1$ are unbiased estimators for β_0 and β_1 for errors following WN, AR(1) and PAR(1) models, we can compare the efficiency of these estimators using (3.19). For simplicity, the comparison will be carried out with the WN model being the reference case. Those relative efficiencies are explicitly given in the following corollary.

Corollary (3.2): The relative efficiency of $\hat{\beta}_0$ and $\hat{\beta}_1$ under AR(1) and PAR(1) models with respect to the WN model are :

$$RE\left[\hat{\beta}_0(AR(1)), \hat{\beta}_0(WN)\right] = \frac{(1-\phi^2)\left[\left(\frac{S_{XX}}{n} + \bar{X}^2\right)\right]}{\left[\left(\frac{S_{XX}}{n} + \bar{X}^2\right) + 2\sum_{t>j}^{n}\sum \text{K}_{tj}\phi^{t-j}\right]},$$

with K_{tj} given by (3.13),

$$RE\left[\hat{\beta}_0(PAR(1)), \hat{\beta}_0(WN)\right]$$

$$= \frac{\frac{\sigma^2}{S_{XX}}\left[\frac{S_{XX}}{n} + \overline{X}^2\right]}{\sum_{\nu=1}^{\omega}\gamma_0(\nu)\sum_{k=1}^{m} c^2_{(k-1)\omega+\nu} + \sum_{\nu=1}^{\omega}\sum_{k=1}^{m}\sum_{j=1}^{(k-1)\omega+\nu-1} c_{(k-1)\omega+\nu}c_j\gamma_{(k-1)\omega+\nu-j}(\nu)I((k-1)\omega+\nu-j\geq 1)}$$

$$RE\left[\hat{\beta}_1(AR(1)), \hat{\beta}_1(WN)\right] = \frac{(1-\phi^2)}{\left[1+2S_{XX}\sum_{t>j}^{n}\sum M_{tj}\phi^{t-j}\right]}$$

and

$$RE\left[\hat{\beta}_1(PAR(1)), \hat{\beta}_1(WN)\right]$$

$$= \frac{\sigma^2/S_{XX}}{\sum_{\nu=1}^{\omega}\gamma_0(\nu)\sum_{k=1}^{m} b^2_{(k-1)\omega+\nu} + \sum_{\nu=1}^{\omega}\sum_{k=1}^{m}\sum_{j=1}^{(k-1)\omega+\nu-1} b_{(k-1)\omega+\nu}b_j\gamma_{(k-1)\omega+\nu-j}(\nu)I((k-1)\omega+\nu-j\geq 1)}.$$

Now, we illustrate Corollary (3.2) through the following example.

Example (3.1): Consider the simple linear regression model $Y_t = \beta_0 + \beta_1 X_t + \varepsilon_t$; t = 1, 2 , …, 100 with X_t = t and

(a) ε_t follow WN with $\sigma^2_\varepsilon = 16$

(b) ε_t follow AR(1) with ϕ = -0.8, -0.6, -0.4, -0.2, 0, 0.2, 0.4, 0.6, 0.8 and $\sigma^2_\varepsilon = 16$

(c) ε_t follow $PAR_4(1)$ with $\prod_{\nu=1}^{4}\phi_1(\nu)$ = 0.8, -0.6, -0.4, -0.2, 0, 0.2, 0.4, 0.6, 0.8 and

$\sigma^2_\varepsilon(\nu) = 16$, $\forall\, \nu = 1, 2, \ldots, 4$.

Using Corollary (3.2) the relative efficiency defined by equation (3.19) of the LS estimates $\hat{\beta}_0$ and $\hat{\beta}_1$ under WN, AR(1) and $PAR_4(1)$ models are computed and summarized in Tables (3.1)-(3.3) and Figures (3.1)-(3.2). To make the comparison more accurate we have fixed $Var(\varepsilon_t)$ for all error models. Besides, we have selected the ϕ's of the $PAR_4(1)$ model such that $\prod_{\nu=1}^{\omega}\phi_1(\nu)$ takes the same value as ϕ for the AR(1) model. We should emphasize that the objective of this example is for illustration and investigation.

Table (3.1): The relative efficiency of $\hat{\beta}_0$ and $\hat{\beta}_1$ under AR(1) model with respect to the WN model

ϕ	$RE(\hat{\beta}_0(AR(1)),\hat{\beta}_0(WN))$	$RE(\hat{\beta}_1(AR(1)),\hat{\beta}_1(WN))$
-0.8	0.123	0.126
-0.6	0.262	0.264
-0.4	0.439	0.440
-0.2	0.674	0.675
0.0	1.000	1.000
0.2	1.484	1.481
0.4	2.278	2.267
0.6	3.812	3.775
0.8	8.004	7.810

Table (3.2): The relative efficiency of $\hat{\beta}_0$ and $\hat{\beta}_1$ under $PAR_4(1)$ model with respect to the WN model

$\phi_1(\nu)$'s	$\prod_{\nu=1}^{4}\phi_1(\nu)$	$RE(\hat{\beta}_0(PAR_4(1)),\hat{\beta}_0(WN))$	$RE(\hat{\beta}_1(PAR_4(1)),\hat{\beta}_1(WN))$
-0.99, 1.45, 0.67, 0.83	-0.8	0.146	0.173
0.97, 0.83, 1.70, -0.44	-0.6	1.462	1.473
1.40, -1.12, 0.38, 0.66	-0.4	0.104	0.114
0.57, 0.70, -0.39, 1.28	-0.2	1.236	1.256
0.00, 0.00, 0.00, 0.00	0.0	1.000	1.000
1.90, 0.84, 0.68, 0.18	0.2	5.897	5.807
1.10, 0.26, 1.50, 0.97	0.4	9.512	9.264
1.60, 1.20, 0.65, 0.45	0.6	12.217	11.697
1.30, 0.50, 0.70, 0.93	0.8	8.633	8.394

Table (3.3): The relative efficiency of $\hat{\beta}_0$ and $\hat{\beta}_1$ under PAR$_4$(1) model with respect to the AR(1) model

$\phi_1(\nu)$'s	$\prod_{\nu=1}^{4}\phi_1(\nu)$	$RE(\hat{\beta}_0(PAR_4(1)),\hat{\beta}_0(AR(1)))$	$RE(\hat{\beta}_1(PAR_4(1)),\hat{\beta}_1(AR(1)))$
-0.99, 1.45, 0.67, 0.83	-0.8	1.187	1.373
0.97, 0.83, 1.70, -0.44	-0.6	5.580	5.580
1.40, -1.12, 0.38, 0.66	-0.4	0.237	0.259
0.57, 0.70, -0.39, 1.28	-0.2	1.834	1.861
0.00, 0.00, 0.00, 0.00	0.0	1.000	1.000
1.90, 0.84, 0.68, 0.18	0.2	3.974	3.921
1.10, 0.26, 1.50, 0.97	0.4	4.176	4.086
1.60, 1.20, 0.65, 0.45	0.6	3.205	3.099
1.30, 0.50, 0.70, 0.93	0.8	1.079	1.075

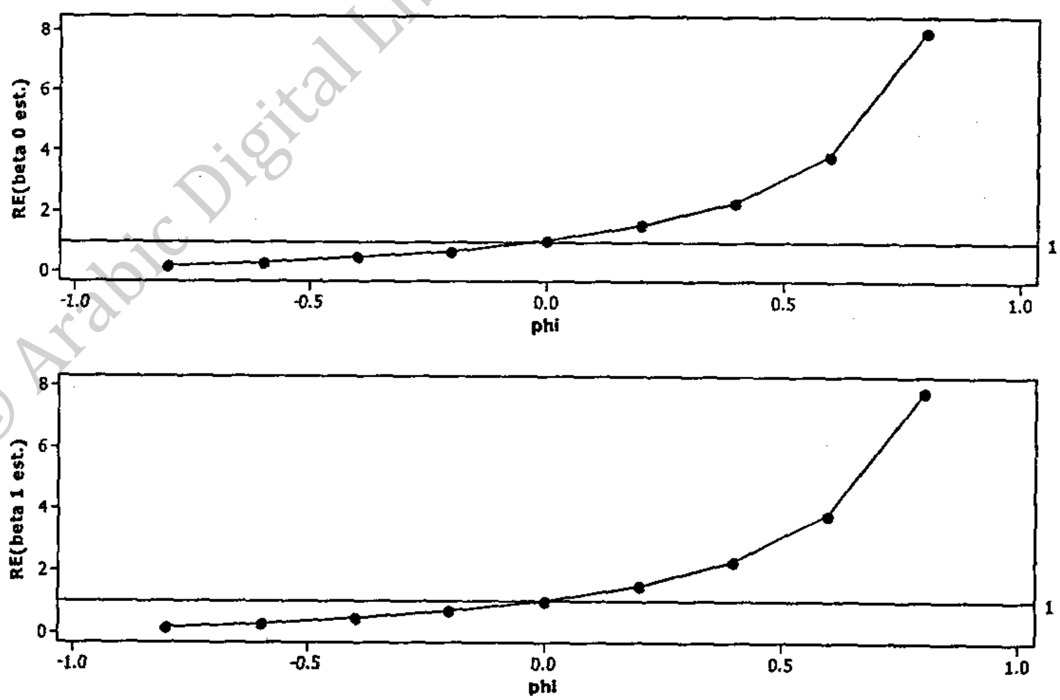


Figure (3.1): The relative efficiency of $\hat{\beta}_0$ and $\hat{\beta}_1$ under AR(1) model with respect to the WN model

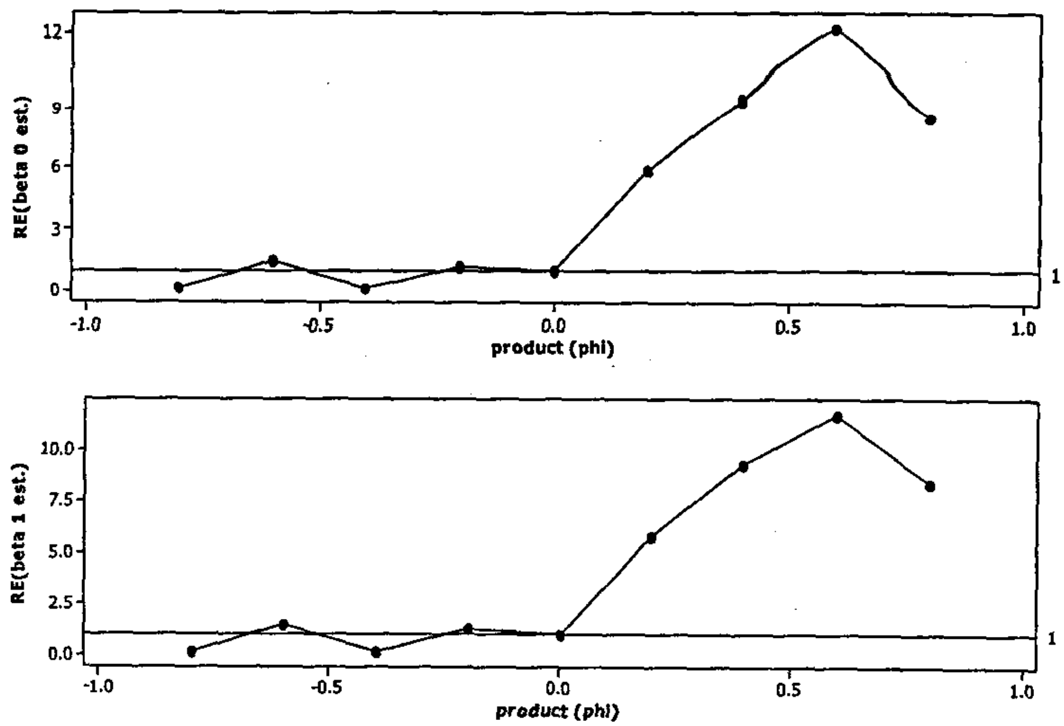


Figure (3.2): The relative efficiency of $\hat{\beta}_0$ and $\hat{\beta}_1$ under $PAR_4(1)$ model with respect to the WN model

In view of Table (3.1) and Figure (3.1), we can see that the LS estimators $\hat{\beta}_0$ and $\hat{\beta}_1$ based on AR(1) model are more efficient than based on WN model when $\phi > 0$. Besides, in view of Tables (3.2), (3.3) and Figure (3.2) we can see that for some cases the RE for $\hat{\beta}_0$ and $\hat{\beta}_1$ were less than one while in other cases was more than one. It is clear here that the main factor is not the value of $\prod_{\nu=1}^{\omega} \phi_1(\nu)$ but possibly the individual values of $\phi_1(\nu)$'s.

Finally, although we show the RE of the LS estimators for various models of errors we recall that if the errors are not WN and are autocorrelated then the well known famous properties of $\hat{\beta}_0$ and $\hat{\beta}_1$ for the simple LR model (e.g. BLUE) are not valid as the assumption of independent errors is no more valid.

The main conclusion here is that when the errors are autocorrelated then the ordinary formulas for $Var(\hat{\beta}_0)$ and $Var(\hat{\beta}_1)$ are not reliable and may underestimate or overestimate the actual variances of those estimators.

3.5 Power of D-W test with errors following various PAR models

In the previous chapter we have investigated the power of D-W test for the generalized regression model when errors follow the WN and AR(1) models. Here, we will use Monte-Carlo simulation to examine the power of D-W test for errors following various PAR models. We will carry out simulation based on the following models:

(1) $PAR_4(1)$ with $\left|\prod_{\nu=1}^{4}\phi_1(\nu)\right|$: 0.1, 0.5, 0.9 and $\sigma_a^2(\nu)$: 1, 10, 50, 40.

(2) $PAR_4(2)$ with $\phi_1(\nu)$: -0.1, 0.8, 0.95, 1.1 and $\phi_2(\nu)$: 0.8, 0.4, -0.7, 0.3 and

$\sigma_a^2(\nu)$: 1, 64, 4, 9.

(3) The varying orders $PAR_4(2,1,0,2)$ with $\phi_1(\nu)$: -0.1, 0.8, 0, 1.1, $\phi_2(\nu)$: 0.8, 0, 0, 0.3 and $\sigma_a^2(\nu)$: 1, 64, 4, 9.

All of the PAR models above are chosen to be periodic stationary. Model (1) is periodic stationary if $\left|\prod_{\nu=1}^{\omega}\phi_1(\nu)\right| < 1$. For the periodic stationarity of Models (2) and (3) see (Al-Quraan, 2010, p. 54-62).

It is worth mentioning that Albertson et al. (2002) have investigated the power of the D-W test when errors in regression models follow the PAR(1) model. Although we believe that there is some overlapping between their work and this research, we emphasize that they have considered only PAR(1) model of errors with constant variance of errors. In this study, we let those variances, $\sigma_a^2(\nu)$, vary. Moreover, we have studied the power of D-W test for other PAR models as the PAR(2) and varying PAR models. In addition, the model PAR(1) is also investigated.

As far as the simulation-work is concerned; 2000 repetitions each of realization length n (100) pairs of data (X,Y) are simulated as the same steps in chapter two but the different in step (2) generate a realization of length n { $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n$ } from one of the models above.

The simulation results for Model (1) are presented in Table (3.4), while the results for Models (2) and (3) are presented in Table (3.5).

Table (3.4): The power of D-W test, errors following $PAR_4(1)$ model (Model(1))

$\Pi\phi_1(\nu)$	$\phi_1(1)$ ($\rho_1(1)$)	$\phi_1(2)$ ($\rho_1(2)$)	$\phi_1(3)$ ($\rho_1(3)$)	$\phi_1(4)$ ($\rho_1(4)$)	Two-sided	Positive auto-correlated
0.1	-0.68 (-0.980)	-0.60 (-0.692)	-0.50 (-0.296)	-0.49 (-0.498)	0.0000	1.0000
	0.68 (0.980)	-0.60 (-0.692)	-0.50 (-0.296)	-0.49 (-0.498)	0.0000	0.5535
	0.68 (0.980)	0.60 (0.692)	-0.50 (-0.296)	-0.49 (-0.498)	0.6020	0.4685
	0.68 (0.980)	0.60 (0.692)	0.50 (0.296)	-0.49 (-0.498)	0.9995	0.9985
	0.68 (0.980)	0.60 (0.692)	0.50 (0.296)	0.49 (0.498)	1.0000	1.0000
0.5	-1.60 (-0.998)	-0.60 (-0.942)	-0.80 (-0.729)	-0.65 (-0.728)	1.0000	0.0000
	1.60 (0.998)	-0.60 (-0.942)	-0.80 (-0.729)	-0.65 (-0.728)	1.0000	0.0000
	1.60 (0.998)	0.60 (0.942)	-0.80 (-0.729)	-0.65 (-0.728)	1.0000	1.0000
	1.60 (0.998)	0.60 (0.942)	0.80 (0.729)	-0.65 (-0.728)	1.0000	1.0000
	1.60 (0.998)	0.60 (0.942)	0.80 (0.729)	0.65 (0.728)	1.0000	1.0000
0.9	-1.30 (-0.999)	-1.10 (-0.993)	-0.90 (-0.962)	-0.70 (-0.944)	1.0000	0.0000
	1.30 (0.999)	-1.10 (-0.993)	-0.90 (-0.962)	-0.70 (-0.944)	1.0000	0.0000
	1.30 (0.999)	1.10 (0.993)	-0.90 (-0.962)	-0.70 (-0.944)	0.2015	0.0000
	1.30 (0.999)	1.10 (0.993)	0.90 (0.962)	-0.70 (-0.944)	1.0000	1.0000
	1.30 (0.999)	1.10 (0.993)	0.90 (0.962)	0.70 (0.944)	1.0000	1.0000

Table (3.5): The power of D-W test, errors following $PAR_4(2)$ (Model(2)) and $PAR_4(2,1,0,2)$ (Model(3))

Model	n	Two-Sided	Positive auto-correlated
$PAR_4(2)$ ($\rho_1(\nu)$): 0.953, 0.803, 0.940, 0.976)	30	0.9995	1.0000
	50	1.0000	1.0000
	100	1.0000	1.0000
$PAR_4(2,1,0,2)$ ($\rho_1(\nu)$): 0.198, 0.172, 0.000, 0.495)	30	0.0595	0.1315
	50	0.1390	0.2375
	100	0.3595	0.5595

3.6 Discussion

It can be seen from Table (3.4) that the significancy of D-W test when errors follow PAR(1) is mainly affected by the magnitudes of seasonal autocorrelations $\rho_1(\nu)$. When all of these autocorrelations are positive and relatively close to 1 then both of the two-sided and positive autocorrelated test were highly significant. Else, if some or all of the autocorrelations are negative then the power of the two-sided test is larger than the power of the positive autocorrelated test. Nearly similar comments apply for the $PAR_4(2)$ and varying order PAR_4 models in Table (3.5). The p-values of D-W test for the $PAR_4(2)$ model were very close or equal to one for both two-sided and right-tail tests. This is attributed to the large positive first lag seasonal correlations for this model. For the varying orders PAR_4 model, the p-value of the right-tail test was larger than those of the two-sided test whereas all p-value are relatively small. This is due to the relatively small (but non-negative) first lag autocorrelations. For this model the power increases as n increases. In general, it seems that the type and orders of the model have less effect on significany of D-W test. The main factor here is rather the signs and magnitudes of $\rho_1(\nu)$; v=1, ..., 4.

CHAPTER 4

Generalization of Cochrane-Orcutt Procedure

4.1 Introduction

In this chapter, we will explain the Cochran-Orcutt Procedure when errors terms are autocorrelated. We will generalize the Cochrane-Orcutt procedure if the errors of the regression model follow the PAR(1) model and we will give an application of the proposed method using Monte-Carlo simulation. Furthermore, we will compare between LS method and Cochrane-Orcutt method via Bias and MSE.

4.2 Cochrane-Orcutt procedure

When the error terms are autocorrelated, then the parameters estimation of the regression model is not straightforward. Under the assumption that the errors follow the AR(1) model given by (2.1), the simple regression model in (1.1) is renamed as the generalized simple linear regression model (Kutner et al., 2005, p. 484). In this case, the model (1.1) can be rewritten as (Kutner et al., 2005, p. 491):

$$Y_t' = \beta_0' + \beta_1' X_t' + a_t, \quad t = 1,2,\ldots, n \tag{4.1}$$

where

$$Y_t' = Y_t - \rho Y_{t-1}, \qquad X_t' = X_t - \rho X_{t-1}, \tag{4.2}$$

$$\left.\begin{aligned} \beta_0' &= \beta_0(1-\rho), \\ \beta_1' &= \beta_1 \end{aligned}\right\} \tag{4.3}$$

and

$$a_t = \varepsilon_t - \rho\varepsilon_{t-1},\ t = 1,2,...,n. \tag{4.4}$$

with $\{a_t\}$ being uncorrelated. Thus, (4.1) is a standard simple linear regression model.

Therefore, the estimation of β_0 and β_1 starts by estimating ρ, then estimating β'_0 and β'_1 in (4.1) and finally obtaining estimates of β_0 and β_1 using (4.3). In fact, there are several methods for estimating ρ in this situation including Cochrane-Orcutt procedure and Hildreth-Lu procedure. In this chapter we will only consider the Cochrane-Orcutt procedure. For more details on this issue see Kutner et al. (2005).

The Cochrane-Orcutt procedure involves an iteration of three steps (Kutner et al., 2005, p. 492):

1. Estimation of ρ. This is accomplished by noting that the autoregressive error process assumed in model (1.1) can be viewed as a regression through the origin:

$$\varepsilon_t = \rho\varepsilon_{t-1} + a_t,\ t = 1, 2, ..., n.$$

Since ε_t and ε_{t-1} are unknown, we use the residuals e_t and e_{t-1} obtained by ordinary least square as the response and predictor variables and estimate ρ by fitting a straight line through the origin. We know that the moment estimator of the slope ρ is:

$$\hat{\rho} = \frac{\sum_{t=1}^{n} e_{t-1}e_t}{\sum_{t=2}^{n} e_{t-1}^2}. \tag{4.5}$$

2. Fitting of transformed model (4.1). Using the estimate $\hat{\rho}$ in (4.5), we next obtain the transformed variables Y_t' and X_t' in (4.2) and use ordinary least squares with these transformed variables to yield the fitted regression function as follow:

$$\hat{Y}_t' = \hat{\beta}_0' + \hat{\beta}_1' X_t'$$

3. Test for need to iterate. The D-W test is then employed to test whether the error terms for the transformed model are uncorrelated. If the test indicates that they are uncorrelated, the procedure terminates. Then $\hat{\beta}_0$ and $\hat{\beta}_1$ are obtained based on $\hat{\beta}_0'$ and $\hat{\beta}_1'$ in the previous step and using (4.3). Besides, an estimate of σ_ε^2 is given by (Kutner et al., 2005, p. 487):

$$\hat{\sigma}_\varepsilon^2 = \frac{\hat{\sigma}_a^2}{1-\hat{\rho}^2}.$$

where $\hat{\sigma}_a^2$ is the sample variance of residuals obtained from the fitted regression model in the previous step.

4. If the D-W test in step (3) above is significant then steps (1)-(3) are repeated for Y' and X' in place of Y and X. Continue until D-W test indicates that error terms are uncorrelated.

4.3 Generalization of Cochran-Orcutt procedure for errors following PAR(1)

Consider the generalized simple linear regression model with error terms following the zero-mean $PAR_\omega(1)$ model, that is:

$$\left.\begin{aligned} Y_t &= \beta_0 + \beta_1 X_t + \varepsilon_t,\ t = 1, 2, \ldots, n \\ \varepsilon_t &= \phi_1(\nu)\varepsilon_{t-1} + a_t \end{aligned}\right\} \quad (4.6)$$

where ν=1, 2, ..., ω denotes the season, $\phi_1(\nu)$ is the AR parameter of season ν with $\left|\prod_{\nu=1}^{\omega}\phi_1(\nu)\right|<1$ and { a_t } is a zero-mean white noise process with periodic variances $\sigma_a^2(\nu)$ and independent of ε_t.

In section (4.2) we have reviewed the Cochrane-Orcutt procedure for the estimation of the parameters of the generalized regression model with errors following AR(1). Now we will generalize this procedure for the model (4.6) above with errors following the $PAR_\omega(1)$ model. Since Y_t and X_t are seasonal time series with period ω, we can rewrite (4.6) as:

$$\left.\begin{aligned} Y_{k,\nu} &= \beta_0 + \beta_1 X_{k,\nu} + \varepsilon_{k,\nu}, \\ \varepsilon_{k,\nu} &= \phi_1(\nu)\varepsilon_{k,\nu-1} + a_{k,\nu} \end{aligned}\right\} \quad (4.7)$$

where the time k denotes the year and ν=1, 2, ..., ω denotes the season.

Proposition (4.1): The generalized regression model (4.7) is equivalent to:

$$Y'_{k,\nu} = \beta'_0(\nu) + \beta'_1(\nu)X'_{k,\nu} + a_{k,\nu}, \quad (4.8)$$

with

$$\left.\begin{aligned} Y'_{k,\nu} &= Y_{k,\nu} - \phi_1(\nu)Y_{k,\nu-1} \\ X'_{k,\nu} &= X_{k,\nu} - \phi_1(\nu)X_{k,\nu-1} \\ \beta'_0(\nu) &= \beta_0(1-\phi_1(\nu)) \\ \beta'_1(\nu) &= \beta_1 \end{aligned}\right\} \quad (4.9)$$

Proof. The result is straightforward by substituting for $Y_{k,\nu}$ and $Y_{k,\nu-1}$ form (4.7) in

$$Y'_{k,\nu} = Y_{k,\nu} - \phi_1(\nu)Y_{k,\nu-1}$$

which gives

$$Y'_{k,\nu} = (\beta_0 + \beta_1 X_{k,\nu} + \varepsilon_{k,\nu}) - \phi_1(\nu)(\beta_0 + \beta_1 X_{k,\nu-1} + \varepsilon_{k,\nu})$$

$$= \beta_0(1-\phi_1(\nu)) + \beta_1(X_{k,\nu} - \phi_1(\nu)X_{k,\nu-1}) + (\varepsilon_{k,\nu} - \phi_1(\nu)\varepsilon_{k,\nu-1}). \quad \#$$

The transformed model in (4.8) is a generalized regression model with errors following a seasonal white noise process with periodic coefficients. To estimate the parameters of this model we note that (4.8) defines a standard regression model for each season separately. That is, for instance to estimate $\beta'_0(1)$, $\beta'_1(1)$ and $\sigma^2_a(1)$ we will use the data for $Y'_{k,1}$ and $X'_{k,1}$ only.

Thus, we summarize the extended Cochrane-Orcutt procedure for errors following $PAR_\omega(1)$ model in the following steps:

(1) Regress Y_t on X_t using ordinary LS method and obtain the residuals $\{e_t\}$.

(2) Test by using D-W test for autocorrelation among residuals. If residuals are not autocorrelated then the procedure terminates.

(3) Estimate $\phi_1(\nu)$ by regressing $Y_{k,\nu}$ on $X_{k,\nu}$ for each season $\nu = 1, 2, \ldots, \omega$ separately. Then obtain the residual for each model $e^*_{k,\nu}$, then obtain $\phi_1(\nu)$ using:

$$\hat{\phi}_1(\nu) = \frac{\sum_{k=1}^{m} e^*_{k,\nu-1} e^*_{k,\nu}}{\sum_{k=1}^{m} (e^*_{k,\nu-1})^2}, \quad (4.10)$$

(4) Compute $Y'_{k,\nu}$ and $X'_{k,\nu}$ using (4.9) and the estimates in (4.10). Then regress $Y'_{k,\nu}$ on $X'_{k,\nu}$ for data in each season v, separately. This gives $\hat{\beta}'_0(\nu)$, $\hat{\beta}'_1(\nu)$ and

$$\hat{\sigma}^2_a(\nu) = \frac{\sum_{k=1}^{m} (e'_{k,\nu})^2}{m-2}.$$

(5) Apply D-W test on $\{e'_{k,\nu}\}$ for each season v= 1, 2, ..., ω. If none of cases is significant then the procedure terminates. Else, if in some seasons the D-W test was significant then we apply the ordinary Cochrane-Orcutt procedure on those seasons until the D-W test is found insignificant for all seasons.

(6) Using $\hat{\beta}'_0(\nu)$, $\hat{\beta}'_1(\nu)$ and (4.9) find $\hat{\beta}_0$ and $\hat{\beta}_1$ which are unbiased estimator of β_0 and β_1. Denote them by $\hat{\beta}_{0\nu}$ and $\hat{\beta}_{1\nu}$; v= 1, 2, ..., ω.

(7) In (6) we will get ω estimates of β_0 and β_1 thus we propose to estimate β_0 and β_1 by the average of these estimates, i.e.

$$\tilde{\beta}_0 = \frac{1}{\omega}\sum_{\nu=1}^{\omega}\hat{\beta}_{0\nu} \quad \text{and} \quad \tilde{\beta}_1 = \frac{1}{\omega}\sum_{\nu=1}^{\omega}\hat{\beta}_{1\nu} \tag{4.11}$$

(8) For the estimation of the variances of $\{\varepsilon_t\}$, (4.7) gives

$$\sigma_\varepsilon^2(\nu) = \phi_1^2(\nu)\sigma_\varepsilon^2(\nu-1) + \sigma_a^2(\nu); \nu = 1, 2, ..., \omega \tag{4.12}$$

Thus, replacing $\phi_1(\nu)$ and $\sigma_a^2(\nu)$ with their estimates obtained above we have a system of ω equations which can be solved for $\sigma_\varepsilon^2(\nu)$ as (3.6) and (3.7).

Finally, we can use a test of periodically autocorrelated errors in step (2) above proposed by McLeod (1995). This test is helpful in distinguishing the naturel of autocorrelation in errors whether is it ordinary autocorrelation or periodic autocorrelation. This test is defined as follows; let the first lag sample autocorrelation for season v for errors $\{e_t\}$ be:

$$r_1(\nu) = \frac{C_1(\nu)}{\sqrt{C_0(\nu)C_0(\nu-1)}},$$

where $C_1(v)$ is the sample seasonal autocovariance of season v and lag 1 defined as:

$$C_1(v) = \frac{\sum_{j=0}^{m-1}(e_{j\omega+v} - \bar{e}_v)(e_{j\omega+v-1} - \bar{e}_{v-1})}{m-1}$$

and

$$C_o(v) = \frac{\sum_{j=0}^{m-1}(e_{j\omega+v} - \bar{e}_v)^2}{m-1},$$

where $\bar{e}_v$ is the sample mean of data in season v and m is the number of years of data.

Then

$$L = N\sum_{v=1}^{\omega}(r_1(v))^2,$$

is asymptotically distributed as Chi-square of ω degrees of freedom under the assumption that there is no autocorrelation of first lag for all seasons. Thus, if $L > \chi^2_{\omega,\alpha}$ then we conclude that the errors are periodically autocorrelated.

4.4 An illustration of the proposed method

In this section we will give an application of the proposed method above using Monte-Carlo simulation. We consider the model:

$$Y_t = 2 + 5X_t + \varepsilon_t \tag{4.13}$$

where, $X_t = u_t + (\cos(2\pi t/4)+50)$ with u_t is uniform (0, 1) and $\{ \varepsilon_t \}$ follows the zero-mean $PAR_\omega(1)$ model:

$$\varepsilon_{k,v} = \phi_1(v)\varepsilon_{k,v-1} + a_{k,v},$$

with $\phi_1(1) = -0.9, \phi_1(2) = 0.6, \phi_1(3) = 0.3, \phi_1(4) = -0.8$ and $\{a_{k,v}\}$ is a seasonal WN process normally distribution with mean zero and variances $\sigma_a^2(1) = 100, \sigma_a^2(2) = 1,$ $\sigma_a^2(3) = 1$ and $\sigma_a^2(4) = 10.$

Therefore, the model in (4.13) is a generalized regression model with $\{\varepsilon_t\}$ following $PAR_\omega(1)$ model. Besides, X_t is considered as a non-random seasonal time series with period 4. This model is simulated using an R-code written by the author with a realization of length n=15 years. The time series plot of Y_t is sketched in Figure (4.1). This figure shows an apparent seasonality with increasing linear trend.

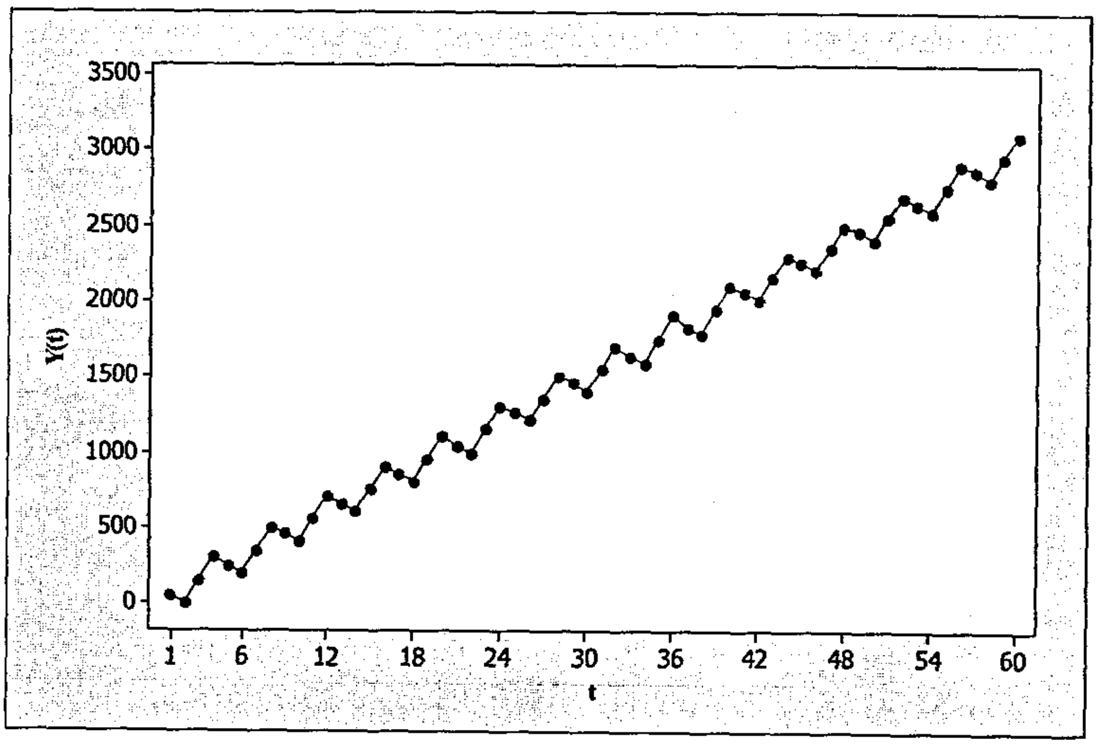


Figure (4.1): The time series plot of Y(t)

Now, following the steps of the proposed modified Cochrane-Orcutt in the previous section:

(1) A simple LR model is fitted of Y on X which gave $\hat{Y} = 1.183 + 50.035X$.

(2) The residuals $\{e_t\}$ in step (1) are computed and the D-W test (two-sided) is applied. It is found that p-value of the test is 0.004 which is significant at α = 0.05.

(3) Y_t and X_t are subdivided as $\{Y_{k,1}, X_{k,1}\}$, $\{Y_{k,2}, X_{k,2}\}$, $\{Y_{k,3}, X_{k,3}\}$ and $\{Y_{k,4}, X_{k,4}\}$ with k=1, …, 30. Then, $Y_{k,v}$ is regressed on $X_{k,v}$ (no-intercept) and gave: $\hat{\phi}_1(1) = -1.812$, $\hat{\phi}_1(2) = 0.590$, $\hat{\phi}_1(3) = 0.298$ and $\hat{\phi}_1(4) = -0.727$.

(4) $Y'_{k,v}$ and $X'_{k,v}$ are obtained using (4.9). Regressing $Y'_{k,v}$ on $X'_{k,v}$ for each season v separately gave:

$$\begin{array}{lll} \hat{\beta}'_0(1) = 7.593 & , \hat{\beta}'_1(1) = 49.990 & , \hat{\sigma}^2_a(1) = 99.173 \\ \hat{\beta}'_0(2) = 1.175 & , \hat{\beta}'_1(2) = 49.977 & , \hat{\sigma}^2_a(2) = 0.817 \\ \hat{\beta}'_0(3) = 2.157 & , \hat{\beta}'_1(3) = 49.962 & , \hat{\sigma}^2_a(3) = 0.906 \\ \hat{\beta}'_0(4) = 4.961 & , \hat{\beta}'_1(4) = 49.977 & , \hat{\sigma}^2_a(4) = 8.555 \end{array}$$

(5) The residuals $\{e'_{k,v}\}$ for each season v= 1, 2, …, ω are computed and the D-W test (two-sided) is applied for each season. It is found that p-values of all tests are: 0.945, 0.817, 0.295 and 0.408. Thus, all are non-significant so that the iterations terminate.

(6) $\hat{\beta}_{0v}$ and $\hat{\beta}_{1v}$ are obtained using (4.9) for each season v separately and gave:

$$\hat{\beta}_{01}=2.700 \quad , \hat{\beta}_{11}=49.990$$
$$\hat{\beta}_{02}=2.865 \quad , \hat{\beta}_{12}=49.977$$
$$\hat{\beta}_{03}=3.071 \quad , \hat{\beta}_{13}=49.962$$
$$\hat{\beta}_{04}=2.873 \quad , \hat{\beta}_{14}=49.977$$

(7) In (6) we have ω= 4 estimates of β_0 and β_1 thus the estimates of β_0 and β_1 are obtained by the average of these estimates (4.11) which gave $\tilde{\beta}_0=2.877$ and $\tilde{\beta}_1=49.977$

(8) The estimates of the variances of $\{\varepsilon_t\}$ are obtained using (3.7) for various seasons as follows: $\hat{\sigma}_\varepsilon^2(1)=136.226$,$\hat{\sigma}_\varepsilon^2(2)=48.191, \hat{\sigma}_\varepsilon^2(3)=5.173$ and $\hat{\sigma}_\varepsilon^2(4)=11.288$.

4.5 Bias and MSE of the proposed estimator using Monte-Carlo simulation

In this section we will study the estimates of β_0 and β_1 for the LS method and the generalized Cochrane-Orcutt method and compare between them via Bias and MSE using Monte-Carlo simulation. We will focus on the estimates of β_0 and β_1 only.

As far as the simulation-work is concerned; 2000 repetitions each of realization length n (30, 50,100) pairs of data (X,Y) are simulated as follows:

(1) Generate the predictor values $X_t = t + 2\text{Cos}(2\pi t/4)$,t= 1, 2, …, nω.

(2) Generate a realization of length nω $\{ \varepsilon_1, \varepsilon_2, \ldots, \varepsilon_{n\omega} \}$ from the zero mean $PAR_4(1)$ model:

$$\varepsilon_{k,\nu} = \phi_1(\nu)\varepsilon_{k,\nu-1} + a_{k,\nu},$$

with $\phi_1(1)=-0.9, \phi_1(2)=0.6, \phi_1(3)=0.3, \phi_1(4)=-0.8$ and $\{a_{k,\nu}\}$ is a seasonal WN process normally distribution with mean zero and variances $\sigma_a^2(1)=100$, $\sigma_a^2(2)=1$, $\sigma_a^2(3)=1$ and $\sigma_a^2(4)=10$.

(3) Compute $Y_t = 2 + 50X_t + \varepsilon_t$; t=1, 2, 3, ...,nω.

(4) Apply the steps from (1-7) of the generalized Cochrane-Orcutt procedure.

(5) Compute the Bias and MSE of estimates β_0 and β_1 for LS method and Cochrane-Orcutt method as follows:

$$Bias = \frac{1}{2000}\sum_{j=1}^{2000}\left((\tilde{\beta}_0)_j - \beta_0\right)$$

and

$$MSE = \frac{1}{2000}\sum_{j=1}^{2000}\left((\tilde{\beta}_0)_j - \beta_0\right)^2.$$

Table 4.1: The Bias and MSE (in brackets) of estimates of β_0 and $\beta1$

n	LS method		Cochrane-Orcutt method	
	$\hat{\beta}_0$	$\hat{\beta}_1$	$\tilde{\beta}_0$	$\tilde{\beta}_1$
30	0.0206 (3.0510)	0.0003 (0.0006)	-0.0055 (0.6821)	-0.000005 (0.0001)
50	0.0360 (1.8351)	-0.0003 (0.0360)	-0.0163 (0.3137)	0.00008 (0.00002)
100	0.0076 (0.9399)	-0.0001 (0.00001)	-0.0049 (0.1389)	0.000003 (0.000002)

Table (4.1) presents the Bias and MSE (in brackets) of estimates for β_0 and β_1 for both the LS method and the generalized Cochrane-Orcutt procedure. Firstly, we emphasize that the least squares estimates are not valid regardless of bias and MSE because the assumption of simple regression model are not satisfied. From Table (4.1) we notice that the bias and MSE of estimates of β_0 and β_1 for both methods decrease as n increases. Besides, the proposed method estimates dominate the LS estimates both in view of bias and MSE. Finally, the bias and MSE for both methods were much smaller for the estimates of β_1 than estimating β_0.

CHAPTER 5

Application to Real Data

5.1 Introduction

In this chapter, a real data set was analyzed as an application of the generalized Cochrane-Orcutt procedure for estimation of the parameters of the simple linear regression model.

5.2 The data

For applying our generalized Cochrane-Orcutt procedure we used a quarterly time series about the U.S. airline passenger-miles (in millions). The data expand 9 years from 1996 to 2004 (Cryer and Chan, 2008). We denote the data by Y_t; t=1, …, 36.

The time series plot of our quarterly time series Y_t is given in Figure (5.1). We can see that the time series shows an increasing trend and seasonality.

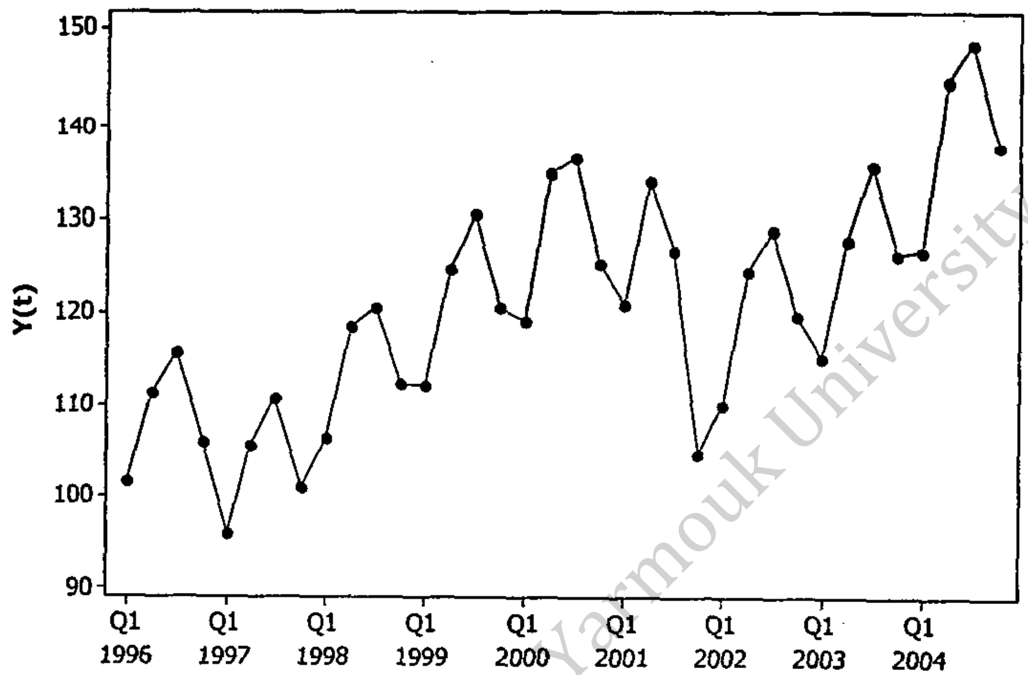


Figure (5.1): The time series plot of quarterly U.S. airline passenger miles (in millions)

5.3 Methodology and analysis

We will apply the generalized Cochrane-Orcutt procedure on Y_t as follows:

(1) A simple LR model is fitted of Y_t on X_t which gave $\hat{Y}_t = 104.608 + 0.866X_t$, where $X_t = t$; t=1, 2, …, 36.

(2) The residuals $\{e_t\}$ in step (1) are computed and the D-W test is applied. It is found that p-value of the test is 0.003 which is significant. To make sure the errors are periodically autocorrelated we will use the McLeod test explained in the previous chapter. It is found that the p-value of this test is 0.00009 which is also highly significant. This means that there is a sufficient evidence that the errors are periodically autocorrelated.

Besides, McLeod (1995) suggested simple graphical methods to investigate periodic autocorrelations in time series. Figure (5.2) shows the parallel box-plots of the residuals $\{e_t\}$ by quarter resulted from the fitted regression model of Y_t vs X_t above. No outliers are detected in this graph. Also the graph shows some differences between the medians and variability of residuals by quarters. This may indicate the presence of seasonality among residuals. The other graphical method is the scatter plots of residuals belonging to successive quarters. This is shown in Figure (5.3). Different nature in the various bivariate relationships in these plots is an indication of periodic autocorrelation in errors.

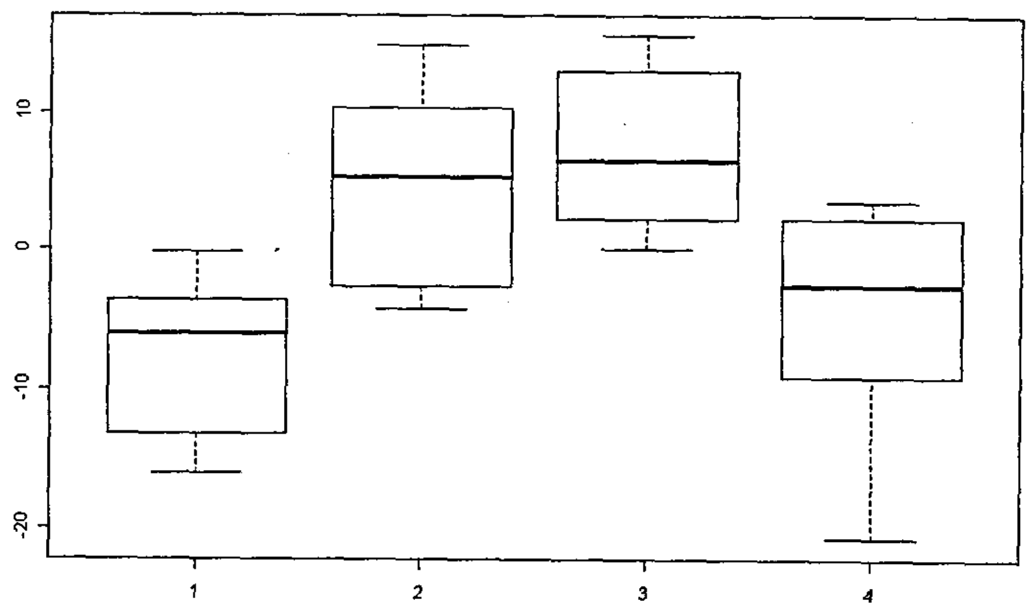


Figure (5.2): Parallel box plot of residuals by quarter

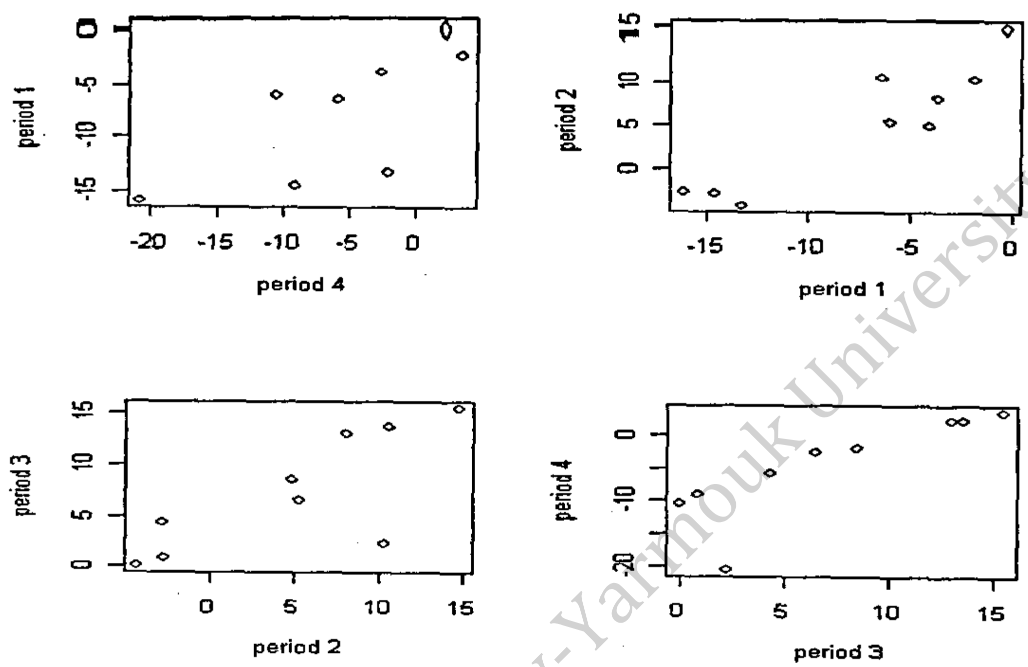


Figure (5.3): Scatter-plots of residuals of consecutive quarters

(3) Y_t and X_t are subdivided by quarter. Then $Y_{k,1}$ is regressed on $X_{k,1}$, $Y_{k,2}$ on $X_{k,2}$, $Y_{k,3}$ on $X_{k,3}$ and $Y_{k,4}$ on $X_{k,4}$; k=1, …,9. The fitted regression models were:

$$\hat{Y}_{k,1} = 99.544 + 0.733 X_{k,1}$$
$$\hat{Y}_{k,2} = 108.495 + 0.925 X_{k,2}$$
$$\hat{Y}_{k,3} = 110.877 + 0.914 X_{k,3}$$
$$\hat{Y}_{k,4} = 100.211 + 0.847 X_{k,4}$$

Thus $\hat{\phi}_1(1) = 0.733$, $\hat{\phi}_2(2) = 0.925$, $\hat{\phi}_3(3) = 0.914$ and $\hat{\phi}_4(4) = 0.847$.

(4) $Y'_{k,v}$ and $X'_{k,v}$ are obtained using (4.9). Regressing $Y'_{k,v}$ on $X'_{k,v}$ for each quarter v separately gave:

$$\hat{\beta}_0'(1)=23.081 \quad , \quad \hat{\beta}_1'(1)=1.081 \quad , \quad \hat{\sigma}_a^2(1)=18.045$$

$$\hat{\beta}_0'(2)=14.339 \quad , \quad \hat{\beta}_1'(2)=3.190 \quad , \quad \hat{\sigma}_a^2(2)=4.135$$

$$\hat{\beta}_0'(3)=10.527 \quad , \quad \hat{\beta}_1'(3)=1.210 \quad , \quad \hat{\sigma}_a^2(3)=18.181$$

$$\hat{\beta}_0'(4)=6.095 \quad , \quad \hat{\beta}_1'(4)=0.584 \quad , \quad \hat{\sigma}_a^2(4)=22.482$$

(5) The residuals $\{e'_{k,v}\}$ for each season v= 1, ..., 4 are computed from the fitted models in (4) and the D-W test is applied for each season. It is found that p-values of all tests are: 0.481, 0.317, 0.273 and 0.419. Thus, all are not significant so that the iterations terminate.

(6) $\hat{\beta}_{0v}$ and $\hat{\beta}_{1v}$ are obtained using (4.11) for each season v separately and gave:

$$\hat{\beta}_{01}=86.468 \quad , \hat{\beta}_{11}=1.081$$

$$\hat{\beta}_{02}=190.195 \quad , \hat{\beta}_{12}=3.190$$

$$\hat{\beta}_{03}=122.848 \quad , \hat{\beta}_{13}=1.210$$

$$\hat{\beta}_{04}=39.937 \quad , \hat{\beta}_{14}=0.584$$

(7) Using above estimates and (4.11) we have $\tilde{\beta}_0=109.862$ and $\tilde{\beta}_1=1.516$.

(8) The estimates of the variances of {ε_t} are obtained using (4.12) and are as follows: $\hat{\sigma}_\varepsilon^2(1)=53.128$, $\hat{\sigma}_\varepsilon^2(2)=49.555$, $\hat{\sigma}_\varepsilon^2(3)=59.607$ and $\hat{\sigma}_\varepsilon^2(4)=65.284$.

CHAPTER 6

Conclusions and Future Work

6.1 Introduction

In this chapter, we summarize our results in this thesis. In addition, we give some research ideas for further studies in the same field of periodically-correlated errors.

6.2 Conclusions

In this thesis we studied the simple linear regression with periodically-correlated errors. Many useful results are found and different models are studied.

Firstly, we investigated if the power of the D-W test is affected by the model of the autocorrelated errors. We have considered WN, AR(1), MA(1) and ARMA(1,1) models with different values of parameters, realization length and the type of test (positive autocorrelated and two-sided test).

For AR(1) model, the test of positive autocorrelated is significant with $\phi > 0$ and the power increases as ϕ is closer to 1 and in two-sided test, the power is symmetric in view of ϕ and the power increases as $|\phi|$ is closer to 1. For the MA(1) and ARMA(1,1) models the power of D-W test was mainly affected by the value of first lag autocorrelation ρ_1.

When the errors follow the $PAR_4(1)$ model and all the first lag seasonal autocorrelations $\rho_1(\nu)$ are positive and relatively close to 1 then both of two-sided and positive autocorrelated test were highly significant. But, if some or all of the autocorrelations are negative then the power of the two-sided test is larger than the power of the positive autocorrelated test. For the PAR(2) and varying orders PAR_4 models the previous conclusion is also true.

Finally, we have generalized the Cochrane-Orcutt procedure of the errors term of the regression model follow the $PAR_4(1)$ model. Then we used this procedure to estimate the regression parameters β_0 and β_1. Also, we apply our procedure on an example and we found that the estimates values are very close the parameters values. Using Monte-Carlo simulation we compare between least squares method and Cochrane-Orcutt procedure for estimation β_0 and β_1 via Bias and MSE, we noticed that the Bias and MSE for both estimates decrease as n increases. Besides, the Bias and MSE for the estimator of β_1 are less than those for β_0 in both methods. A real application on the proposed procedure was also provided.

6.3 Future work

Simple linear regression model with errors following $PAR_4(1)$ model has been investigated. However, we believe that there are several issues regarding regression model with autocorrelated error need further research. For instance, the following two statements are possible ideas for future work:

- Studying the multiple regression models as well as the multivariate regression model with periodically autocorrealted errors.
- Using other procedures for estimation the parameters of regression models in case of autocorrelated errors as the Hildreth-Lu and First Differences procedures (Kutner et al., 2005, p. 495).

References:

1. Albertson, K., Aylen, J. and Lim, K. B. 2002. "The power of durbin Watson test when the errors are PAR(1)", *J. Statist. Comput. Simul.*, Vol. 72(6), pp.507-516.
2. Al-Quraan, Areen 2010. *On Robust Estimation of Seasonal Autocorrelation Function for Periodic Autoregressive Models*. Master thesis, Department of Statistics, Yarmouk University, Jordan.
3. Blattberg, R. 1973. "Evaluation of the power of the durbin-watson statistic for non-first order serial correlation alternatives", *journal Review of Economics & Statistics*, Vol. 55, pp. 508-515.
4. Bowerman, B., O'Connell, R. and Koehler, A. 2005. *Forecasting, Time Series, and Regression*. 4th Edition, Thomson.
5. Box, G., Jenkins, G. and Reinsel, G. 1994. *Time Series Analysis, Forecasting and Control*. 3rd Edition, Prentice-Hall, New Jersey.
6. Cryer, J. and Chan, K. 2008. *Time Series Analysis with application in R*. Springer Publishers.
7. Durbin, J. and Watson, G. S. 1950. "Testing for serial correlation in least-squares regression: I." Biometrika, 37, pp. 409-428.
8. Franses P. and Paap, R. 2004. *Periodic Time Series Models*, Oxford Univ. Press.
9. Fried, R. and Gather, U. 2005. "Robust Trend Estimation for AR(1) Disturbances", *Austrian Journal of Statistics*, Vol. 34, pp.139-151.
10. Gulhan, A. and Pierre, D. 2001. "Efficiency analysis of Ten Estimation Procedures for quantitive Linear Models with Autocorrelated errors ", *J. Statist. Comput. Simul.*, Vol. 69, pp. 257-275.

11. Huitema, B. E. and McKean, J. W. 2007. "An improved portmanteau test for autocorrelated error in interrupted time-series regression model ", *Behavior Research Methods*, Vol. 39 Issue 3, pp. 343-349.

12. Kayode, A. 2008. "Performances of Some Estimators of Linear Model when Stochastic Regressors are Correlated with Autocorrelated Error Terms ", *European Journal of Scientifi Research*, Vol. 20 No.3, pp. 558-571.

13. Kutner, H., Nachtsheim, J., Neter, J. and Li, W. 2005. *Applied Linear Statistical Models*. 5th Edition, McGraw-Hill.

14. McLeod, A.I. 1995." Diagnostic Checking Periodic Autoregression Models With Application ", *Journal of Time Series Analysis 15(2)*, pp. 221-233.

15. Mohammed, N. and Ibazizen, M. 2008. "Bayesian Estimation of the Regression Model with Autocorrelated Errors Under Contamination ", *Communications in Statistics: Theory & Methods*, Vol. 37 Issue 15, pp. 2382-2389.,

16. Obeysekera, J. T. B. and Salas, J. D. 1986. "Modeling of Aggregated Hydrologic Time Series ", *J. Hydrol.*, Vol. 86, pp. 197-219.

17. Ohtani, K. 1990. " On Estimating and Testing in a Linear Regression Model with Autocorrelated errors ", *Journal of Econometrics*, Vol. 44 Issue 3, pp. 333-346.

18. Pagano, M. 1978. " On periodic and multiple autoregressions ". *Annals of Statistics*, 6, pp. 1310-1317.

19. Rohatgi, V. K. 1984. *Statistical Inference* . Johan Wiley & Sons, Inc.

20. Schmidt, P. and Guilkey, D. 1975. " some further evidence on the power of the durbin-eatson and geary tests", *journal Review of Economics & Statistics*, Vol. 57, pp. 379-382.

21. Smadi, A. 2004. "LS estimation of periodic autoregressive models with non-Guassian errors", *Journal of Statistical Computation & Simulation*, Vol. 75 Issue 3, pp. 207-216.

22. Smith, M., Wong, Ch. and Kohn, R. 1998. "Additive nonparametric regression with autocorrelated errors ", *Journal of the Royal Statistical Society: Series B (Statistical Methodology)*, Vol. 60 Issue 2, pp. 21-311.

23. Stocker, T. 2007. "On the asymptotic bias of OLS in dynamic regression models with autocorrelated error ", *Statistical Papers*, Vol. 48 Issue 1, pp. 81-93.

24. Tiao, G. and Grupe, M. 1980. "Hidden periodic autoregressive-moving average models in time series data ". *Biometrika*, 67, 365-373.

25. Yue, F. and Koreisha, S. G. 2004. "Forecasting with serially correlated regression models ", *Journal of Statistical Computation & Simulation*, Vol. 74 Issue 9, pp. 625-649.

26. Ula, T. A. and Smadi, A. A. 1997. "Periodic Stationarity Conditions for Periodic Autoregressive Moving Average Processes as Eigen Value Problems ", *Water Resources Res.*, 33, pp. 1929-1934.

27. Wei, W. 1990. *Time Series Analysis*, Univariate and Multivariate Methods. Addison-Wesley Publishing Company.